

Lecture 5: Cellular Automata Dynamics

Complex Systems 530

Quick intro to dynamical systems concepts

- Since we're going to talk about dynamics of ABM, CA, and other kinds of models, want to just briefly introduce/refresh some terminology

Equilibrium Points

- **Equilibrium Point** - a set of values for the variables such that the model will stay constant as time evolves (i.e. all $dx/dt = 0$)
- Note that all variables must stay constant for the whole system to be at equilibrium

Equilibrium Points

- Examples - population growth, etc.

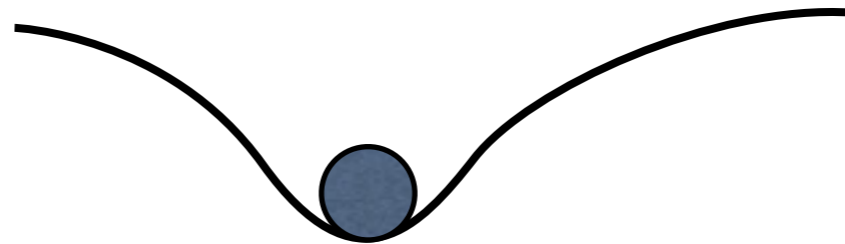
- 1)
$$\frac{dx}{dt} = kx$$

- 2)
$$\frac{dx}{dt} = kx\left(1 - \frac{x}{N}\right)$$

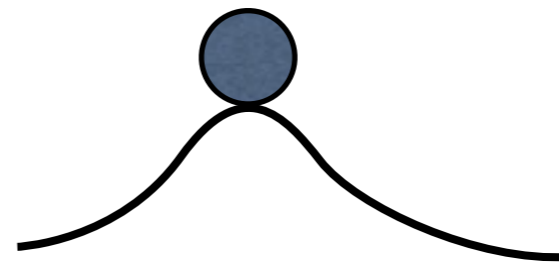
- When are these systems at equilibrium? What do the equilibria represent?

Types of Equilibria

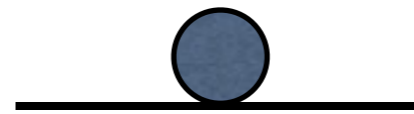
- **Stable**



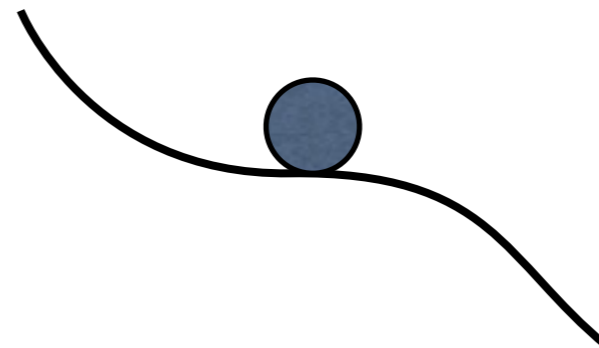
- **Unstable**



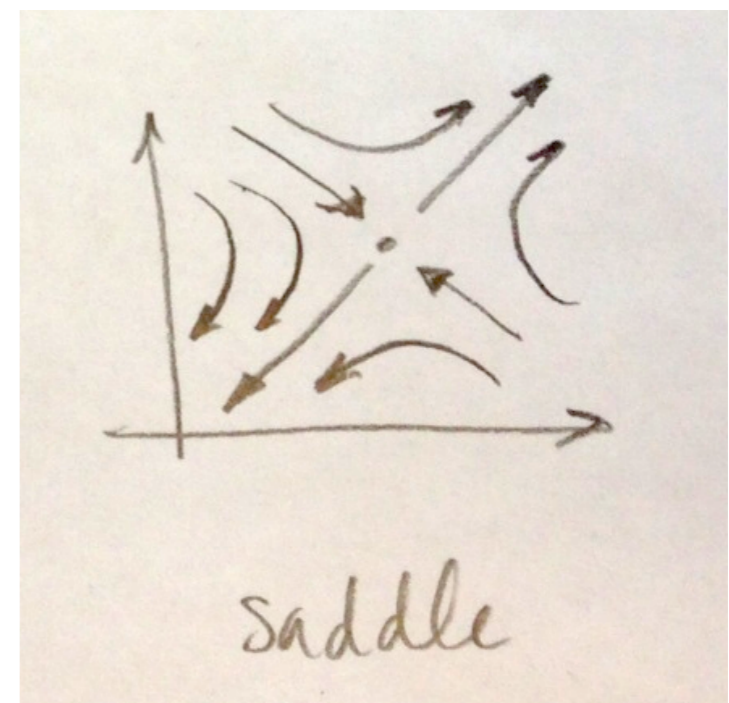
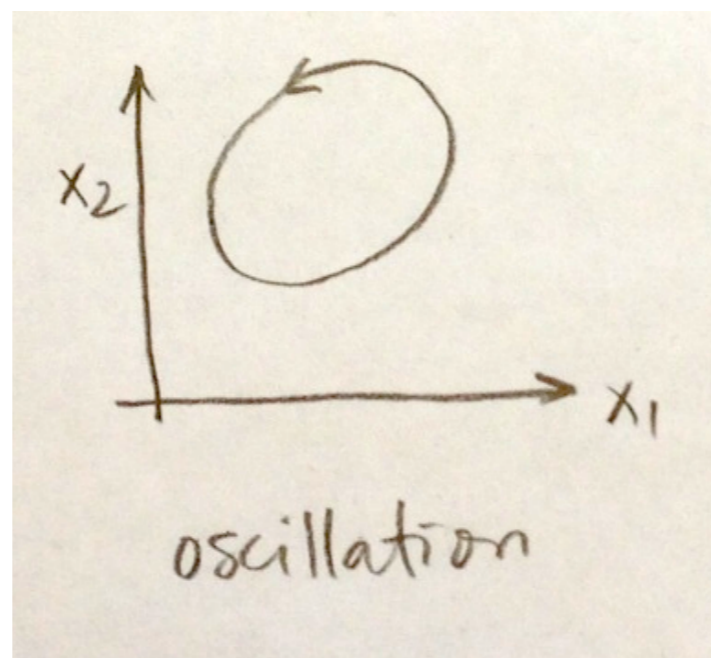
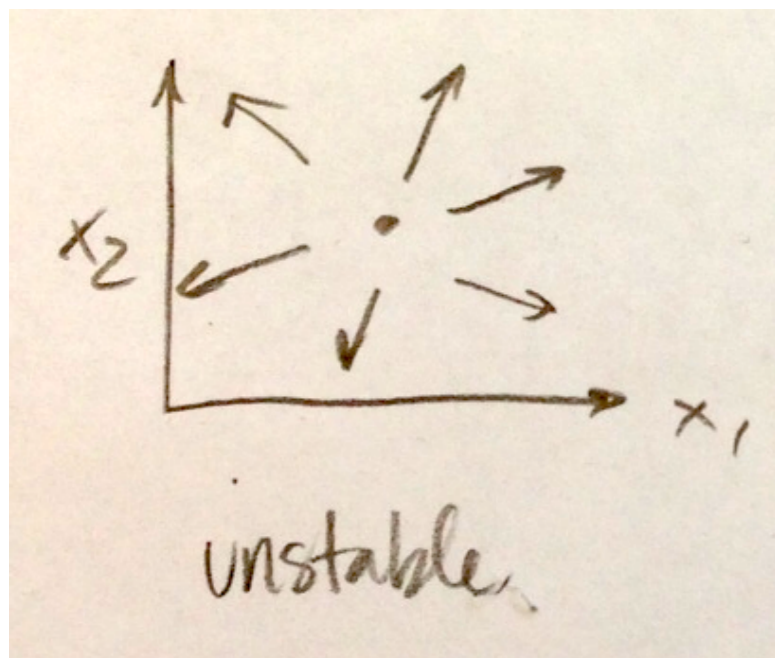
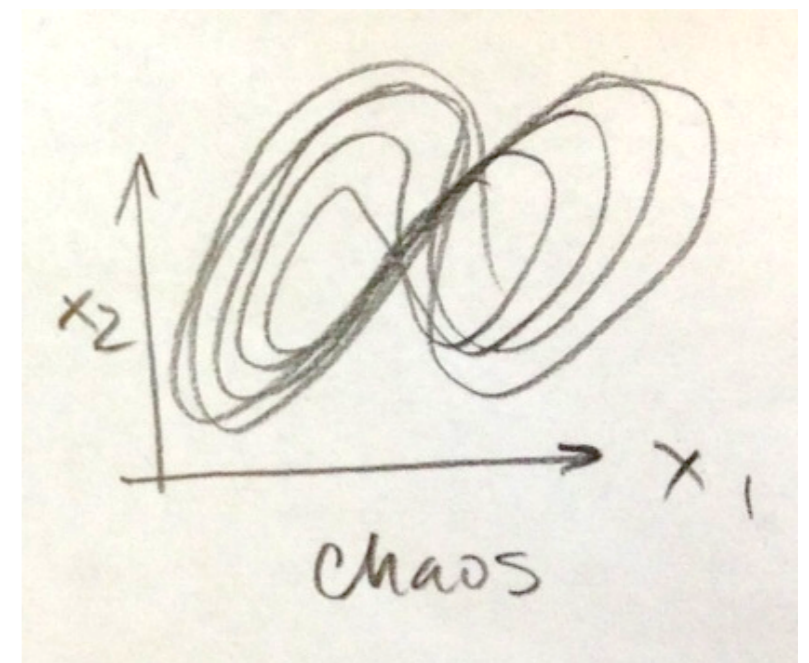
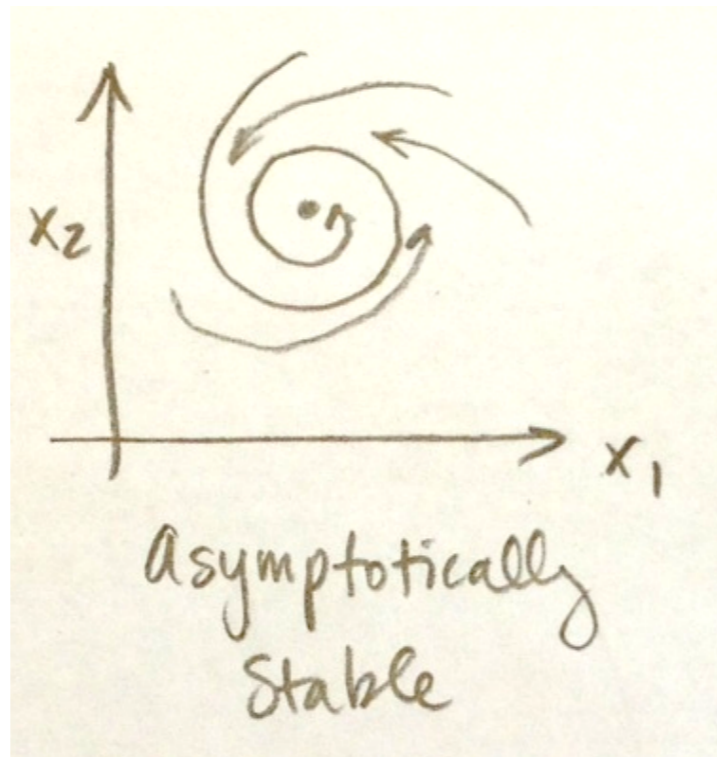
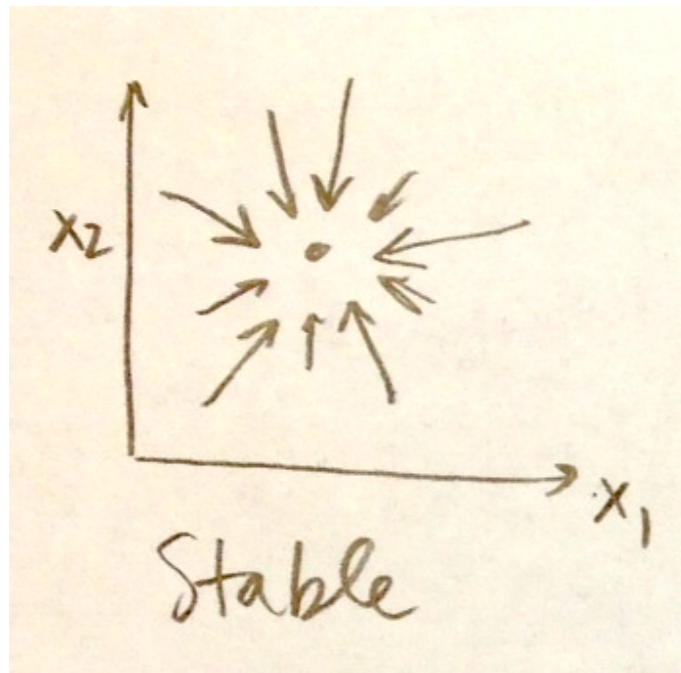
- **Neutral**



- **Saddle**



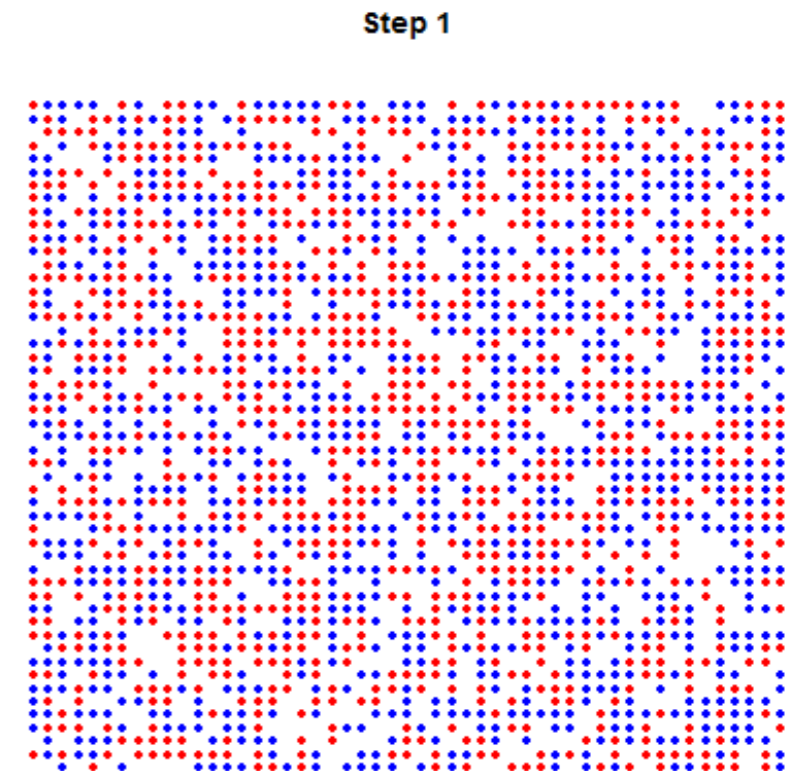
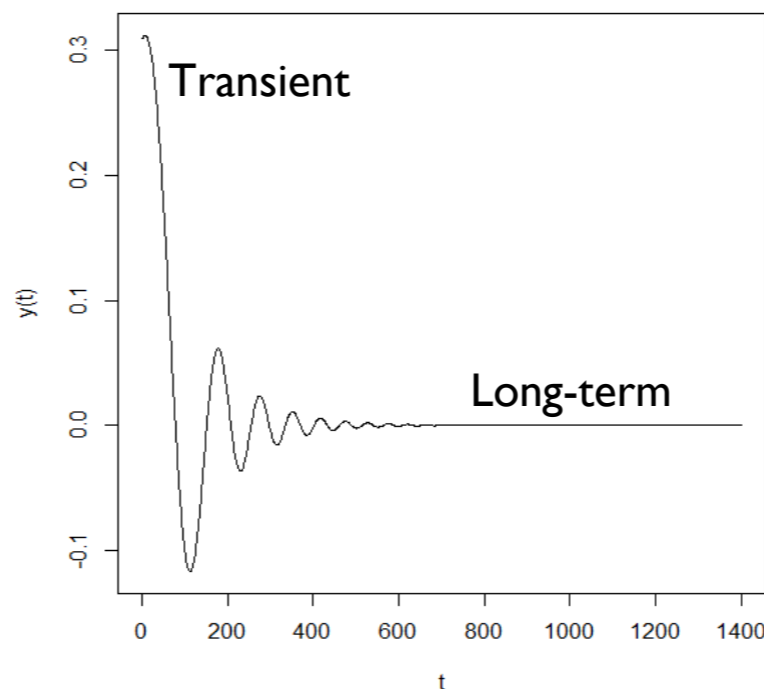
Phase plane/phase space: tells us what the model does for any given initial conditions of the model



and many others...

Transient vs Long-term Behavior

- **Transient** - portion of the model response that dies out/ goes to zero
- **Long-term** - persistent model behavior as $t \rightarrow \infty$
 - Unstable
 - Stable/constant steady state
 - Oscillation
 - Chaos, etc.



Phase transitions/ bifurcations

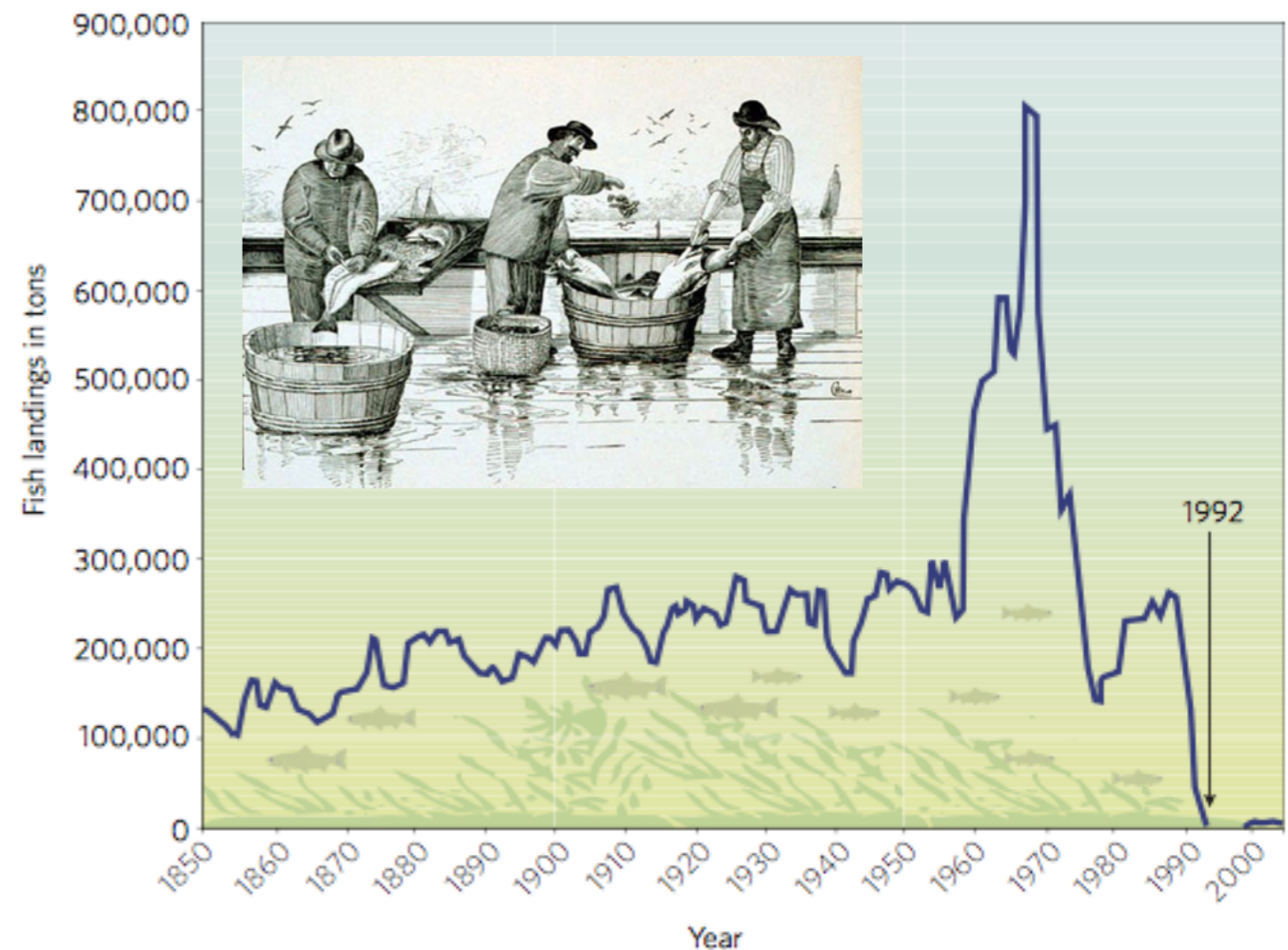
- A phase transition is a “transition of macroscopic properties of a collective system that occurs when its environmental or internal conditions are varied”
- More generally, we often see bifurcations/
qualitative changes in behavior as we move across
parameter space

What are bifurcations?

- A **bifurcation** is a qualitative change in behavior as parameters are varied
- The parameter value where this change happens is called a **bifurcation point**
- Can create or destroy fixed points, change stability, induce oscillations, & more

Qualitative changes in behavior: population collapse

- Advanced fishing trawlers introduced in 50's/60's
- Cod fishery collapse
- 1992 moratorium
- However, still not recovered (only 10-33% of original stock)
- What happened?



Qualitative changes in behavior

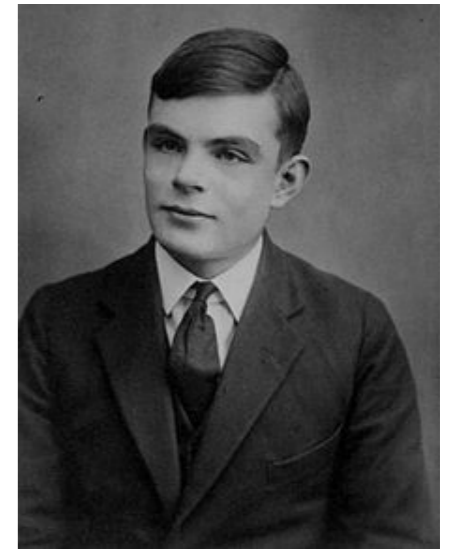
- Development of resistance in bacteria? Bifurcation or just multiple equilibria?
- Onset of cancer—can think of as a bifurcation from controlled growth & death (equilibrium) to uncontrolled growth
- Wide range of other signaling mechanisms controlling cell dynamics can be framed this way (cell cycling, apoptosis, & more)
- Switches between brain states—e.g. sleep, epilepsy

Epileptic Seizure EEG



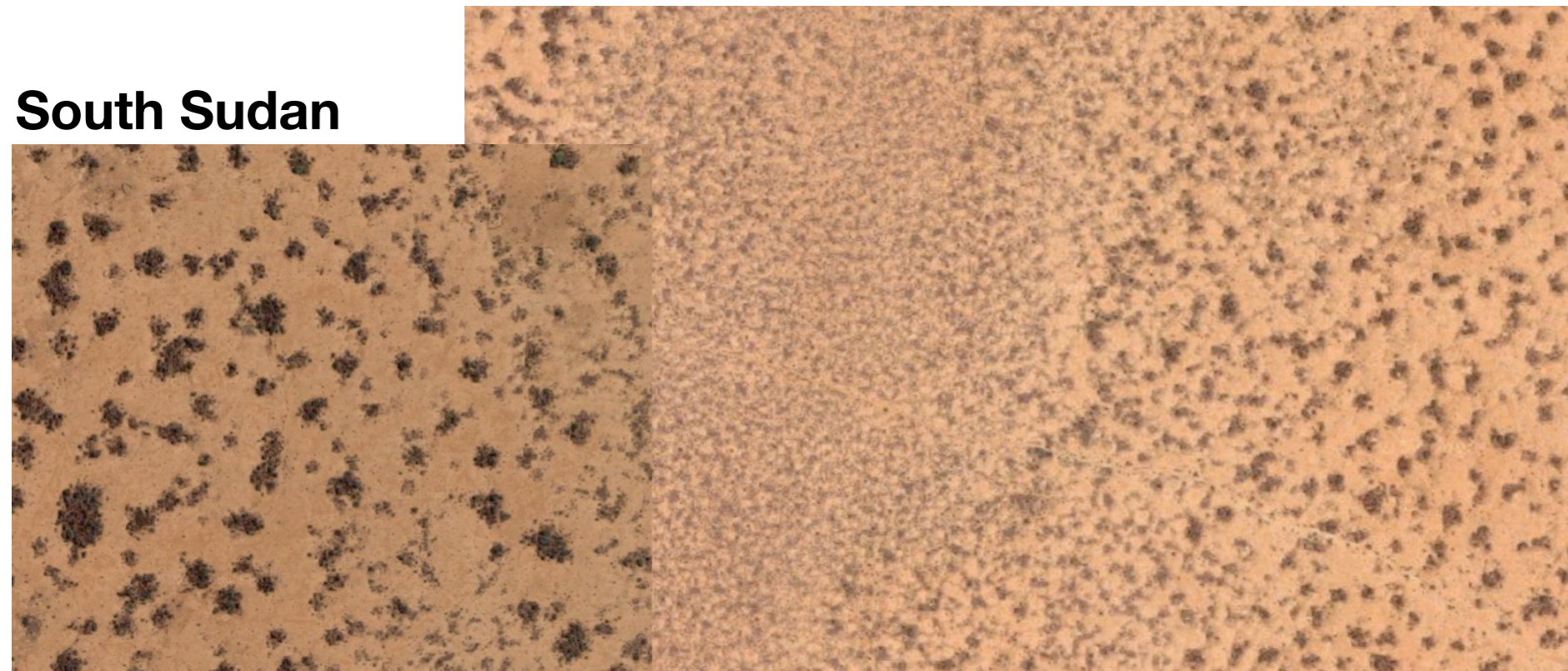
Not just temporal changes: vegetation patterns!

- Pattern formation in vegetation
- Changes in elevation/moisture/etc. can cause surprising changes in plant patterns across space!



Alan Turing

South Sudan



Google Earth

Negev, Israel

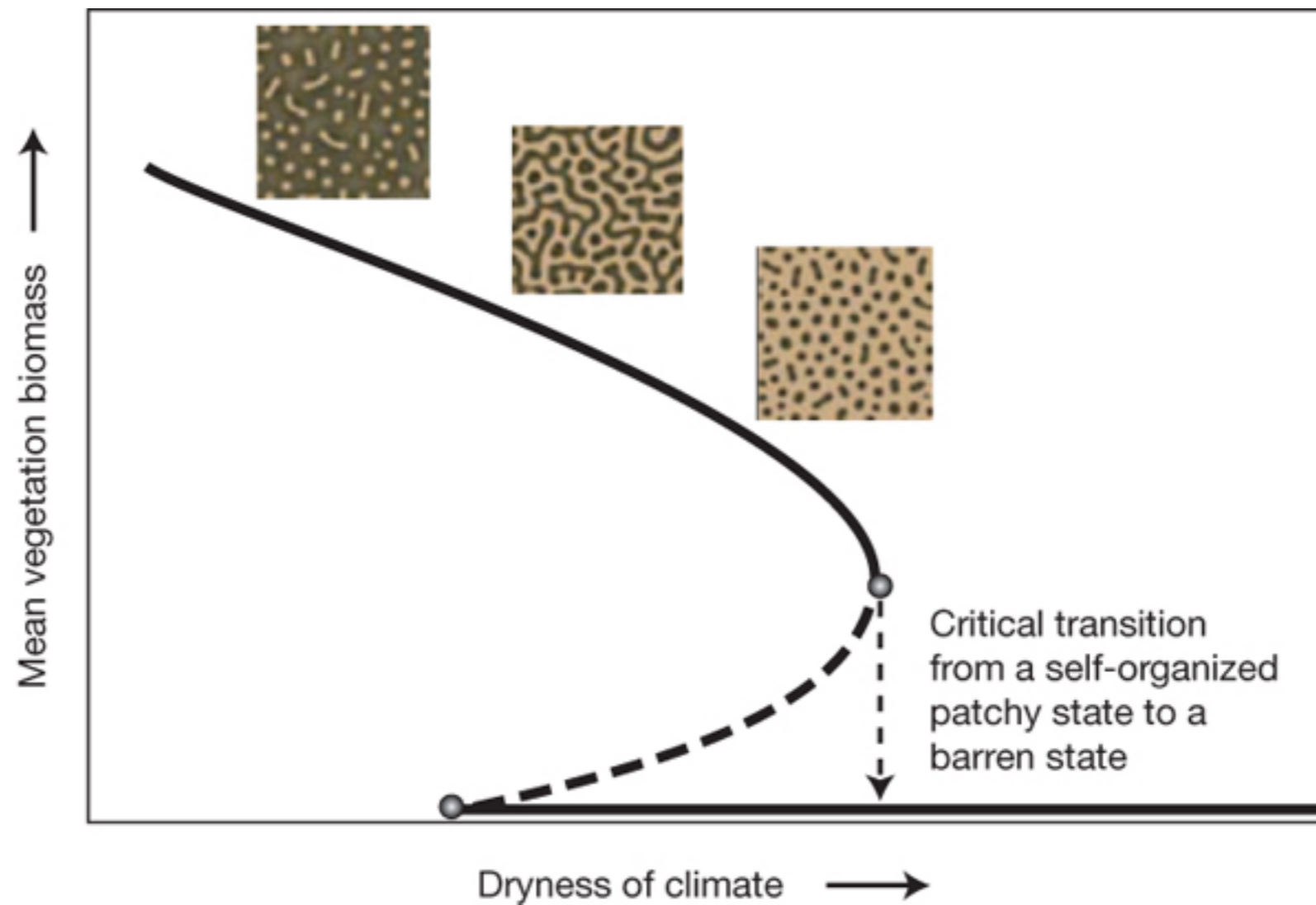


J. von Hardenberg/BIDR/Ben Gurion Univ.

Vegetation patterns



Vegetation patterns



Disease dynamics

- **The most classic bifurcation point in infectious disease epidemiology: $R_0 = 1$**
 - When $R_0 < 1$ the disease-free equilibrium (DFE) is stable (**outbreak dies out**)
 - When $R_0 > 1$, it is unstable (**epidemic!**)
- Basically all intervention efforts & vaccine campaigns are trying to push us across a bifurcation point to eliminate disease

CA & ABM models with phase transitions/bifurcations

- Many examples even if not formally proven to change stability etc. (e.g. Schelling, voting model, etc.)
- Try out together:
 - Forest fire/percolation model
 - Host pathogen model
- Other useful concepts from dynamical systems: basins of attraction, bistability, etc.

Bifurcation vs. multiple equilibria/long-term behaviors

- In a real world setting, can be hard to distinguish!
- In a model setting, bifurcations occur due to changes in parameters
- Whereas changes in the model behavior due to multiple equilibria/multiple long-term behaviors occur due to changes in the initial conditions of the variables

How to explore the space of CA behaviors?

- For simple models, we can examine the **phase space**
- Phase space is the space (in this case a network) of all possible states of the model

CA phase space

- How many different state configurations can we have?
 - D = number of dimensions (1, 2, 3, etc.)
 - L = length in each dimension (number of cells)
 - r = neighborhood radius (how many cells out to consider)
 - k = number of states (binary, more?)

How many different configurations can we have?

- Total cells in the space: L^D
- Each cell can be in one of k states
- Total possible configurations for the system: k^{L^D}
- E.g., a 2D 10x10 binary CA has $2^{10^2} = 1,048,576$ possible configurations

CA rule space

- How many different rules (CAs) can we have?
- Total cells in neighborhood (including self):

$$(2r + 1)^D$$

- Total possible configurations for a single neighborhood (termed situations):

$$k^{(2r+1)^D}$$

- For each situation we map to a resulting state, so total possible rules (CAs) is:

$$k^{k^{(2r+1)^D}}$$

Very big!

Phase space

- Phase space is the space of all possible states of the model—for CA this is discrete, and finite if we have a finite domain
- We can map how one configuration of the model moves to another—forms a network
- Phase space comes from the analogous idea for continuous dynamical systems—there we have a continuous flow from one state to another, for CA we have a directed network

Phase space

- How to map the network of transitions between states?
- We can translate a configuration of space into a binary number, and use this to label each space
- Connect edges from each configuration to the next as we step through time

$$\begin{array}{r} 0\ 0\ \mathbf{1}\ 0\ 0 = 4 \\ 0\ \mathbf{1}\ 0\ \mathbf{1}\ 0 = 10 \\ \mathbf{1}\ 0\ \mathbf{1}\ 0\ \mathbf{1} = 21 \end{array}$$



Phase Space

- We can use the network structure to understand the dynamics of CAs
- Gets tricky for larger grid spaces—many more nodes in the network
- Many of the usual approaches for understanding networks can be used to examine dynamics (cycles, connectedness, etc.)
- Similar to state transition diagram/matrix for Markov models

Phase Space Example

- Binary 1D CA, neighborhood radius 2
- 9 cells in ring arrangement (wrapped boundary)
- 'Majority rule'
- Total possible configurations = $2^9 = 512$

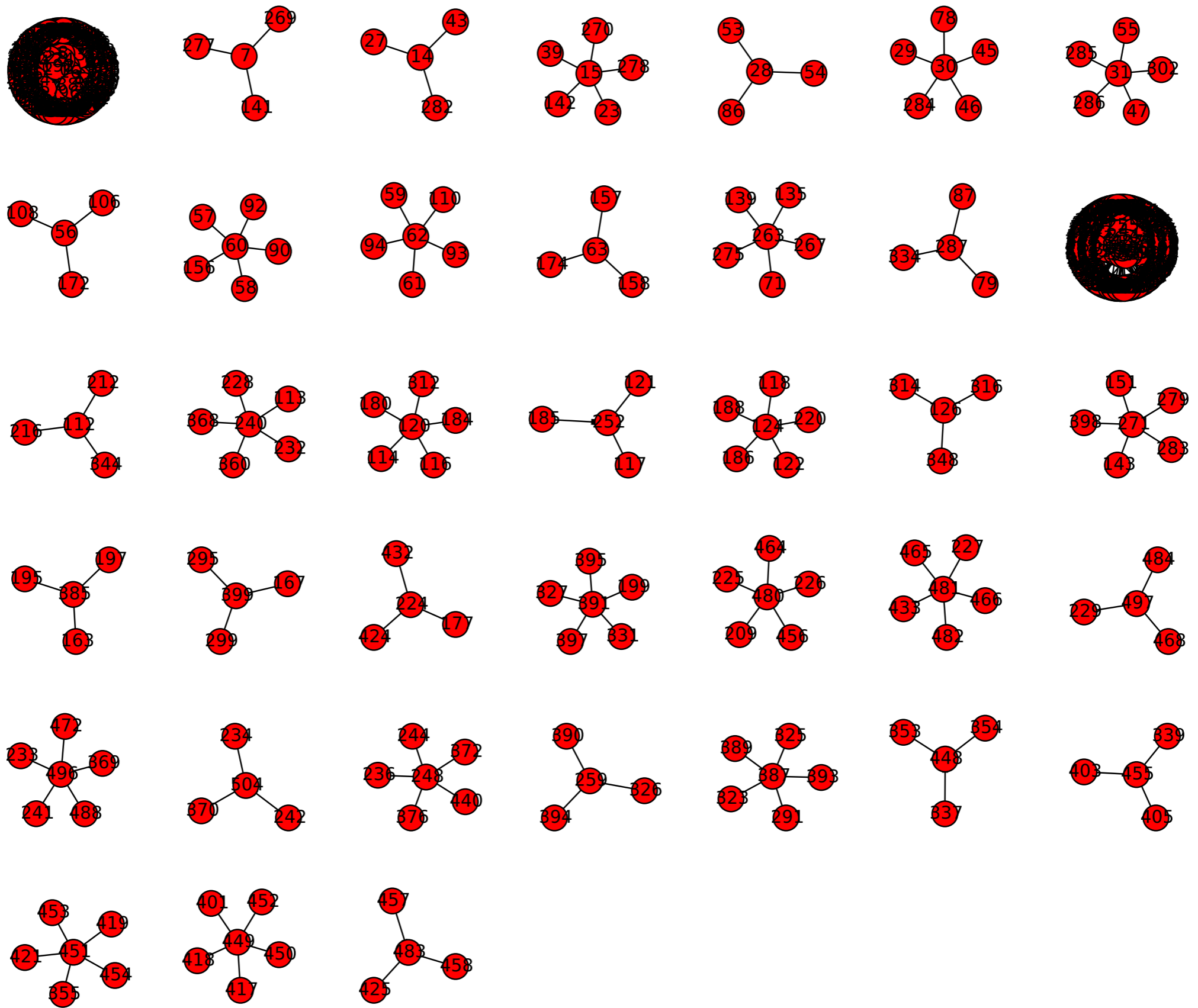


Figure 12.1: Graph-based phase space of the 1-D binary CA model with the majority rule ($r = 2$, $L = 9$) drawn with Code 12.3.

Phase Space Example

- Many different basins of attraction, i.e. network components
- 2 larger basins of attraction—explore with PyCX code
- What is structure overall? What does it look like the majority rule model will do?
- Explore together

Phase Space

- For larger grid sizes, can be much more complicated, networks can become hairball-like
- Some dynamic patterns run for a long time before stabilizing, e.g. the 'rabbit' in Game of Life takes 17,331 steps to stabilize (a very long path in the phase space network)

Phase space exploration

- Code phase space for several 1D CA using example code
- Explore together
- Look for:
 - Attracting subsets, cycles, gardens of eden
 - What do these correspond to dynamically?

Mean-field approximation

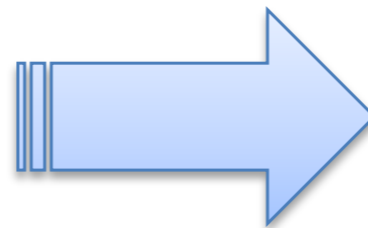
- As CA get more complicated, direct examination of phase space becomes more challenging
- Mean field approximations give one way to understand the dynamics in a very(!) rough way
- Mean field approximation describes the overall average state of the system over time (i.e. how many on/off cells on average)
- Much lower dimension—but also loses most of what makes CA interesting?

Mean-field approximation

Actual State



Mean-field approximation



Approximated State

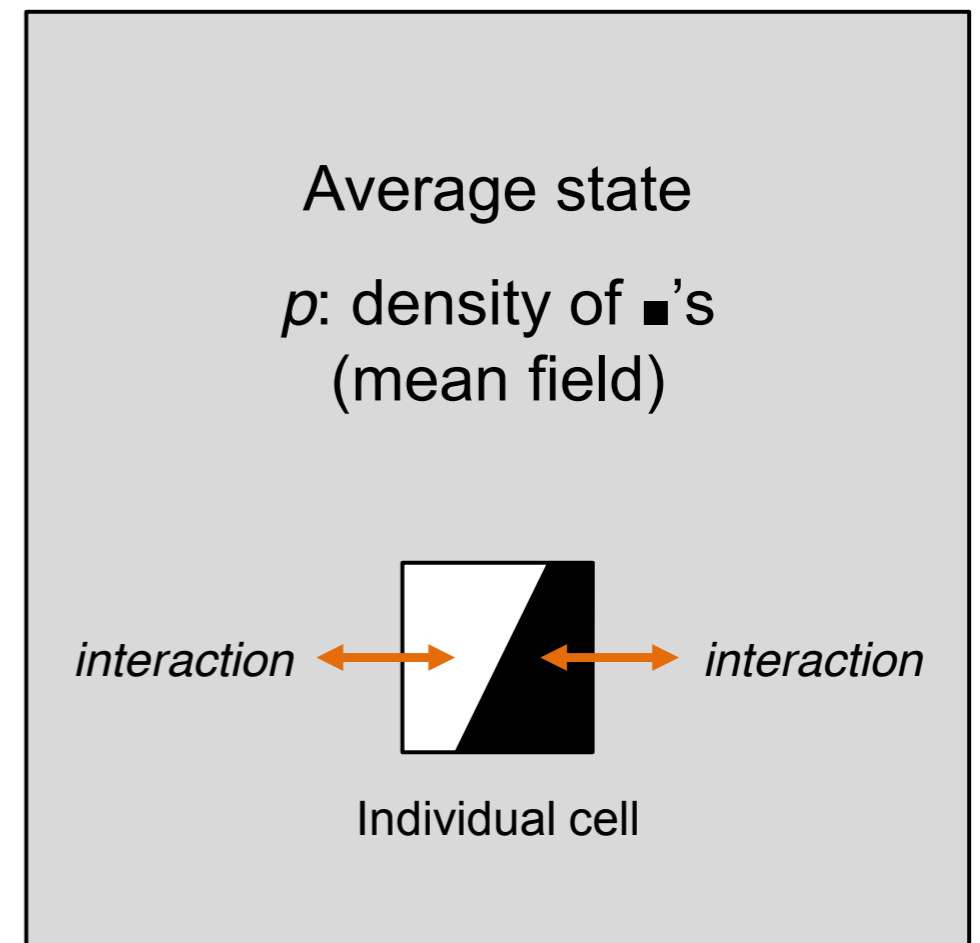


Figure 12.2: Basic idea of the mean-field approximation.

Mean-field approximation

- Consider a 2D binary CA with majority rule
- Let p_t be the density of 1's (on state) in the grid at time t
- We can treat the system probabilistically—work out the probability that a cell would transition on/off given the rules, with no particular knowledge of the exact actual configuration of any given cell

Mean-field approximation

Table 12.1: Possible scenarios of state transitions for binary CA with the majority rule.

Current state	Neighbors' states	Next state	Probability of this transition
0	Four 1's or fewer	0	$(1 - p) \sum_{k=0}^4 \binom{8}{k} p^k (1 - p)^{(8-k)}$
0	Five 1's or more	1	$(1 - p) \sum_{k=5}^8 \binom{8}{k} p^k (1 - p)^{(8-k)}$
1	Three 1's or fewer	0	$p \sum_{k=0}^3 \binom{8}{k} p^k (1 - p)^{(8-k)}$
1	Four 1's or more	1	$p \sum_{k=4}^8 \binom{8}{k} p^k (1 - p)^{(8-k)}$

- $p(\text{state}) \times p(\text{neighbors' states})$

Mean-field approximation

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- $p_{t+1} = p(\text{state is a 1 at next time step})$

Mean-field approximation

$$\begin{aligned} p_{t+1} &= (1-p) \sum_{k=5}^8 \binom{8}{k} p^k (1-p)^{(8-k)} + p \sum_{k=4}^8 \binom{8}{k} p^k (1-p)^{(8-k)} \\ &= \sum_{k=5}^8 \binom{8}{k} p^k (1-p)^{(8-k)} + p \binom{8}{4} p^4 (1-p)^4 \\ &= \binom{8}{5} p^5 (1-p)^3 + \binom{8}{6} p^6 (1-p)^2 + \binom{8}{7} p^7 (1-p) + \binom{8}{8} p^8 + 70 p^5 (1-p)^4 \\ &= 70 p^9 - 315 p^8 + 540 p^7 - 420 p^6 + 126 p^5 \end{aligned}$$

Mean-field approximation

- Gives us a simple, 1-dimensional difference equation that we can use to track the overall probability/density of 1's vs. 0's in the system
- Can determine p_0 from initial conditions and then simulate forward

Cobweb plot

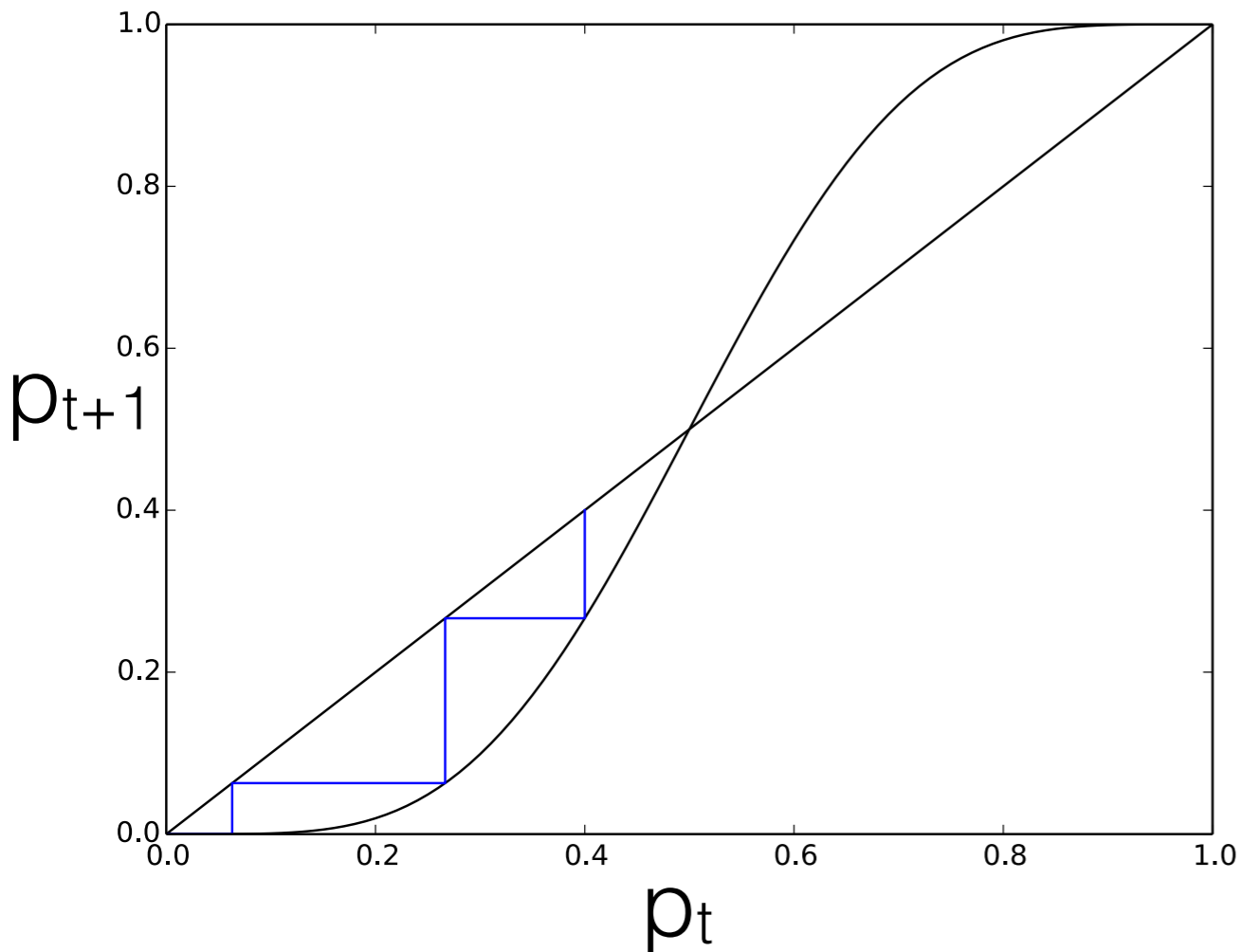


Figure 12.3: Cobweb plot of Eq. (12.10).

- Plots current value vs next value
- Straight line of $y = x$
- Model function plotted as the curve,
$$p_{t+1} = 70p^9 - 315p^8 + 540p^7 - 420p^6 + 126p^5$$
- Where these two intersect, we have an equilibrium point!

Cobweb plot

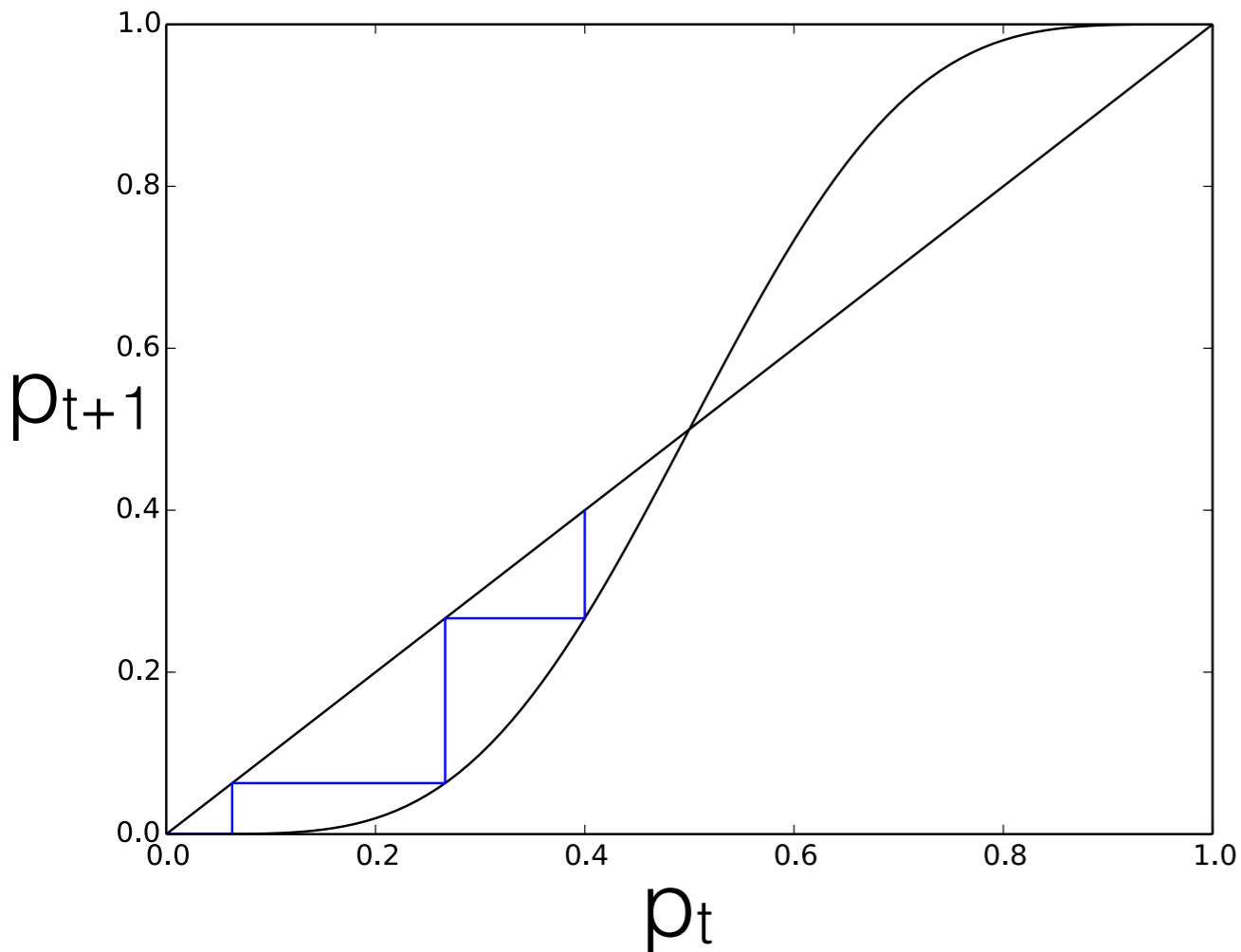


Figure 12.3: Cobweb plot of Eq. (12.10).

- In this case, the cobweb plot shows 3 equilibria
 - All 0 - stable
 - All 1 - stable
 - Half-and-half - unstable
- How true is this to the real CA? Why?

Mean-field approximation

- Does not account for spatial features of the system!
 - It will necessarily be very approximate and represent only the “average” behavior of the system assuming all cells experience a homogeneous ‘neighborhood’
- Is this a good approximation for most CA?
- See also the renormalization group approach for percolation (Sayama Chapter 12)

Extensions to CA

- **Stochastic (probabilistic) CA** - state transitions happen with some probability based on neighboring states (cf. Markov chains)
- **Multi-layer CA** - state values as vectors, e.g. may capture multiple properties or attributes of the agent, or different agents living on the same cell
- **Asynchronous CA** - updates non-simultaneously (e.g. random, ordered, state-triggered)

A note about spaceships & other structures

- Many spaceships and other stable patterns in CA
- An interesting question of whether these are “real”?
- The CA is made of cells, they do all the operations of the model
- The patterns we observe are not actual objects—just persistent patterns that we name and treat as separate entities



A note about spaceships and other structures

- Although, this can be said of a lot of things? (E.g. storms, maybe even people?)
- Doesn't necessarily make the objects in CAs less real because they are composed of cells

For next time...

- Reading
 - Sayama Chapter 12
 - Think Complexity Chapter 7