Lecture 11: Sensitivity analysis

Complex Systems 530
3/12/20

Outline

- Online format
  - Lecture
  - Final project presentations

Sensitivity analysis

- Uncertainty quantification: examines the variation in model outputs & behaviors

- **Sensitivity analysis**: examines which inputs/ parameters drive that variation

Goals

- Capture the frequency/distribution of different outputs/behaviors observed across parameter space as a function of the parameters

- Search for extremes/oddities, i.e. potentially uncommon behaviors that match a criteria (e.g. costly, interesting), illustrate the extreme range of behaviors

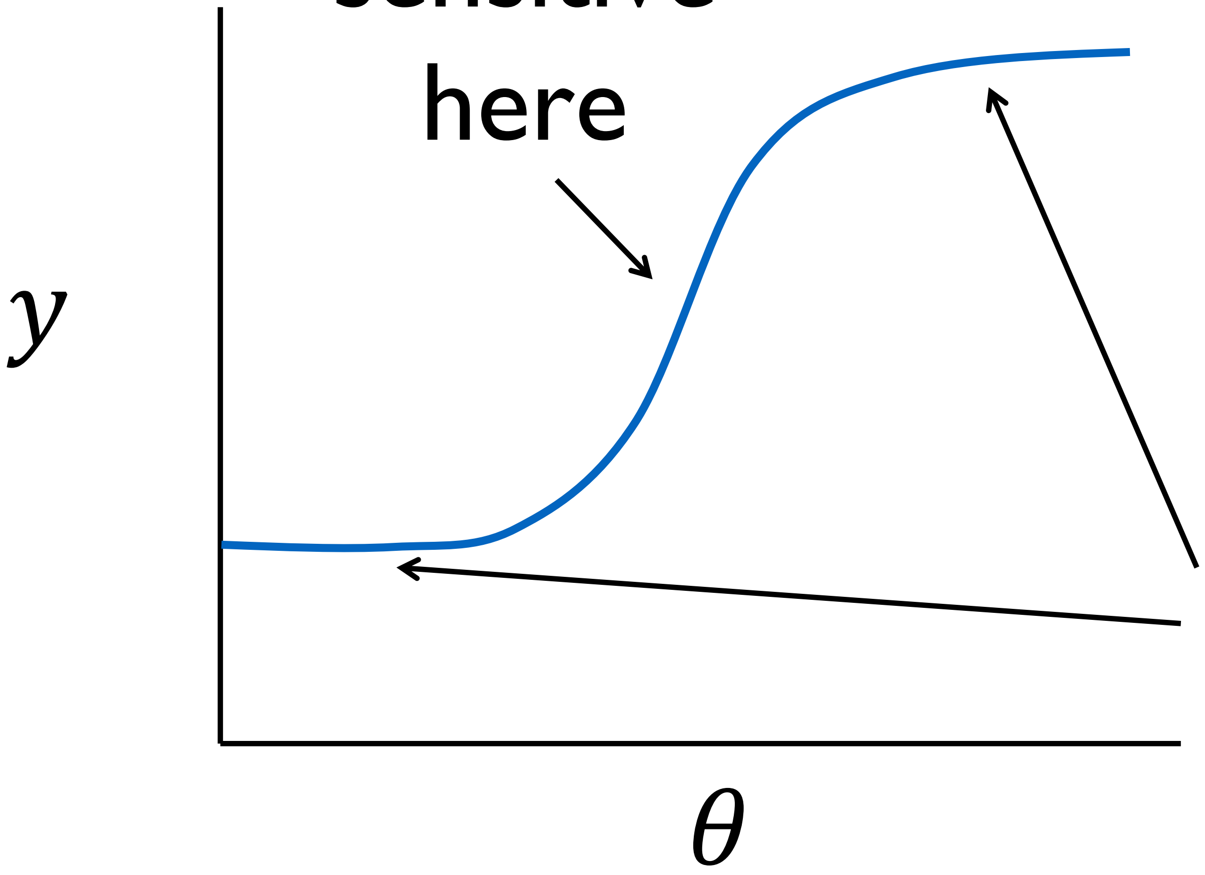
Goals

- Find sensitive/insensitive parameters or parameter combinations
  - Use these to decide what parameters to adjust/ tune/intervene on
  - Reduce model complexity by fixing insensitive parameters

Basic setup

- Adjust a parameter or multiple parameters of interest
- Evaluate the model behavior/output
- Look for trends in how output changes as a function of parameters

Sensitivity is inherently a local attribute



y is very sensitive here

y is not very sensitive here

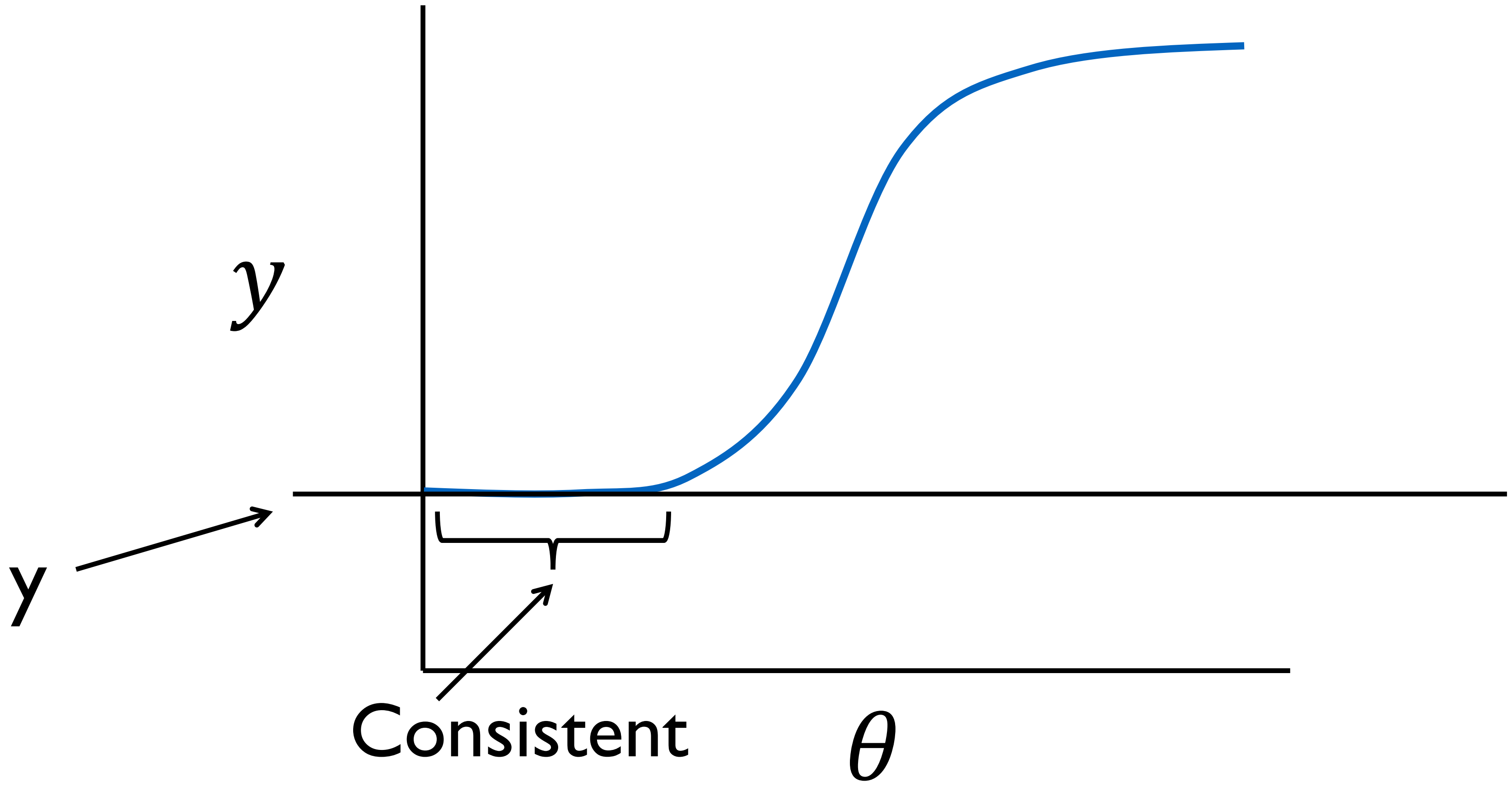
Sensitivity/robustness tradeoff

- How dependent is the model behavior on specific parameter values, initial conditions, inputs?
- Sensitivity - how much the model output changes as a function of the parameter(s)
- Robustness - is the model able to reproduce similar behavior across a range of parameter space?

Sensitivity/robustness tradeoff

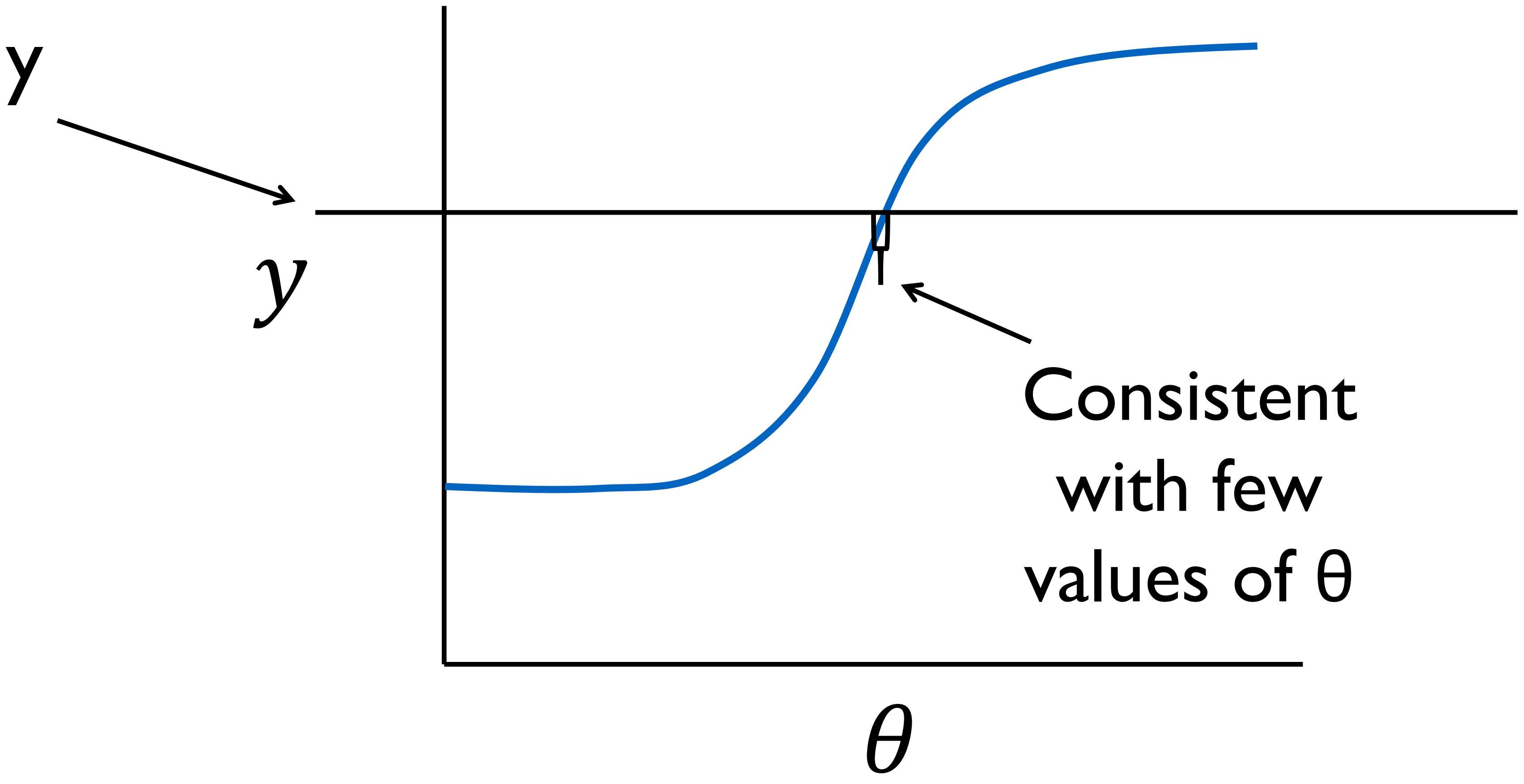
- Sensitivity/robustness tradeoff
- When a behavior is robust, we may have more confidence in it—but, this means we cannot be sure of what parameters generated the behavior
- Similarly, when the output is highly sensitive, we may be better able to infer what parameter conditions must be

Sensitivity/robustness tradeoff



Consistent with many values of θ

Sensitivity/robustness tradeoff



Consistent with few values of θ

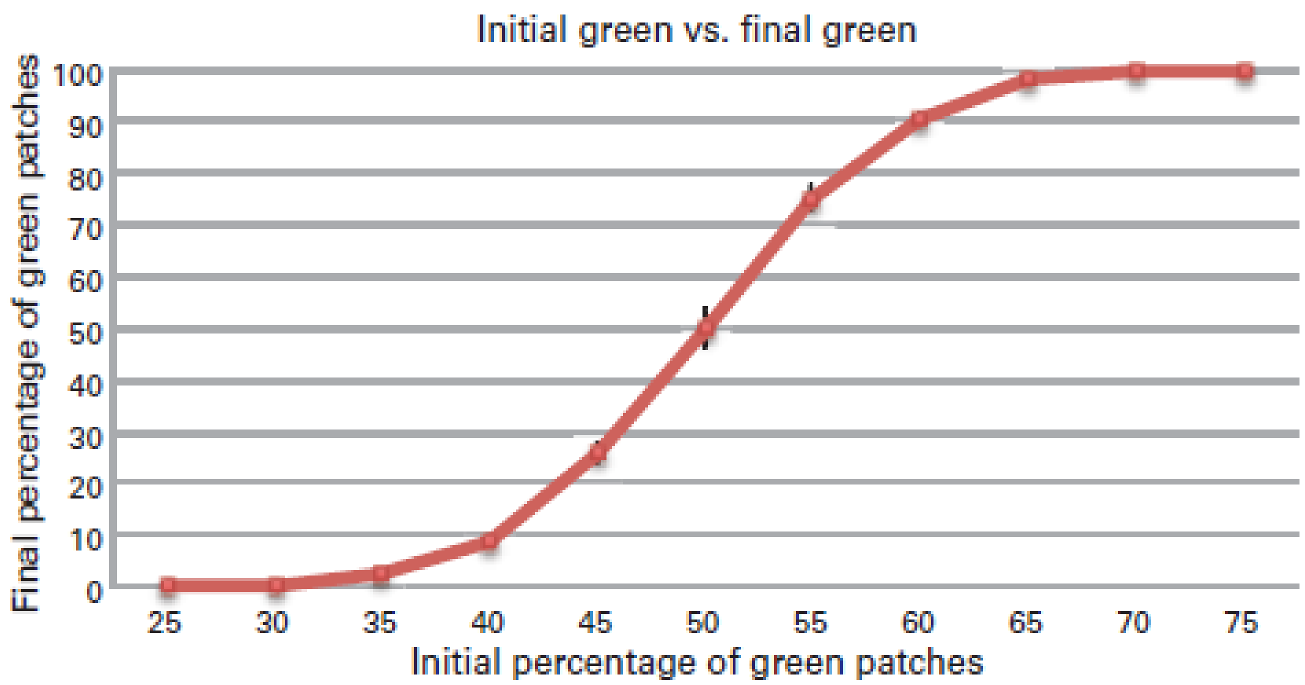
Local Methods

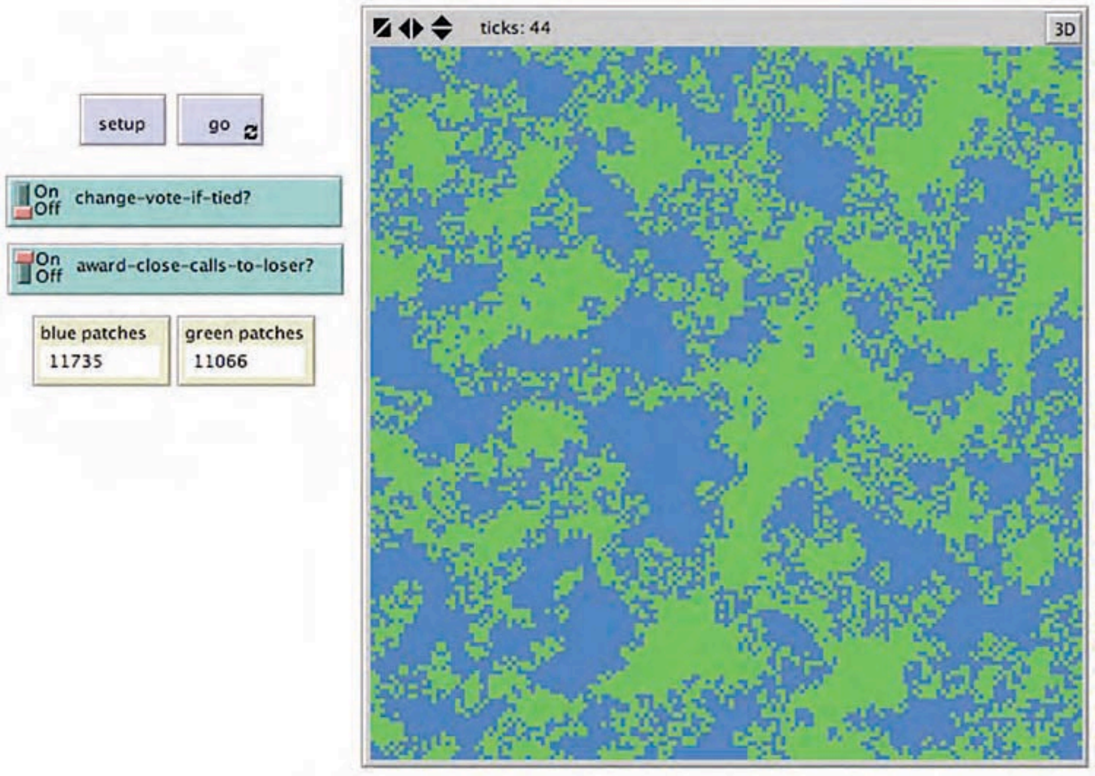
- One-at-a-time approaches
- Derivative methods for local sensitivity

One-at-a-time approach

- Adjust one parameter at a time, fixing the rest to pre-specified values

- Example: voting model initial fraction yes/no





On Off change-vote-if-tied?

On Off award-close-calls-to-loser?

blue patches 11735

green patches 11066

Derivative-based local sensitivity

- Model $\dot{x} = f\left(x,t,p\right)$
- Output $y = g(x,t,p)$
- Output **sensitivity** to parameter variations

$$dy / dp$$

- Meaning depends on magnitude of y and p—often more useful to look at **relative sensitivity**

$$\frac{dy / y}{dp / p} = \frac{dy}{dp}\frac{p}{y}$$

How to calculate local sensitivity?

- Many methods—practically speaking, often done simply by testing small perturbations (e.g. 5% change) of the parameters and seeing how the output changes

$$\frac{dy\,/\,y}{dp\,/\,p} \approx \frac{\Delta y\,/\,y}{\Delta p\,/\,p} = \frac{\%\ \text{change in}\ y}{\%\ \text{change in}\ p}$$

Sensitivity Analysis

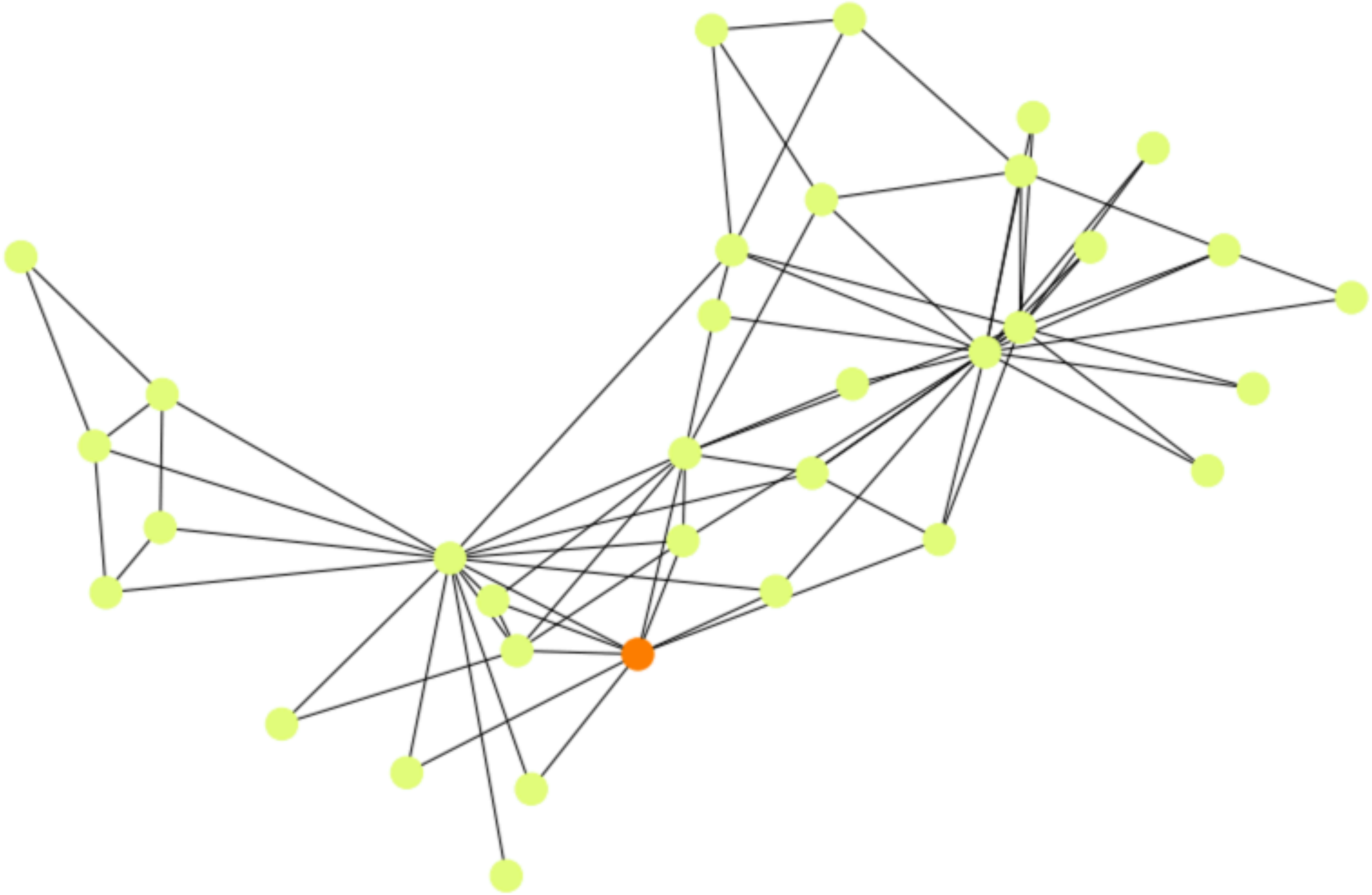
- Sensitivity is inherently a local attribute
- But often we want to know about **global sensitivity** over a wide range of values of θ
- Especially when we have a lot of parameters and want to drop some insensitive ones
- Helps to know where to allocate resources in general for a variety of scenarios

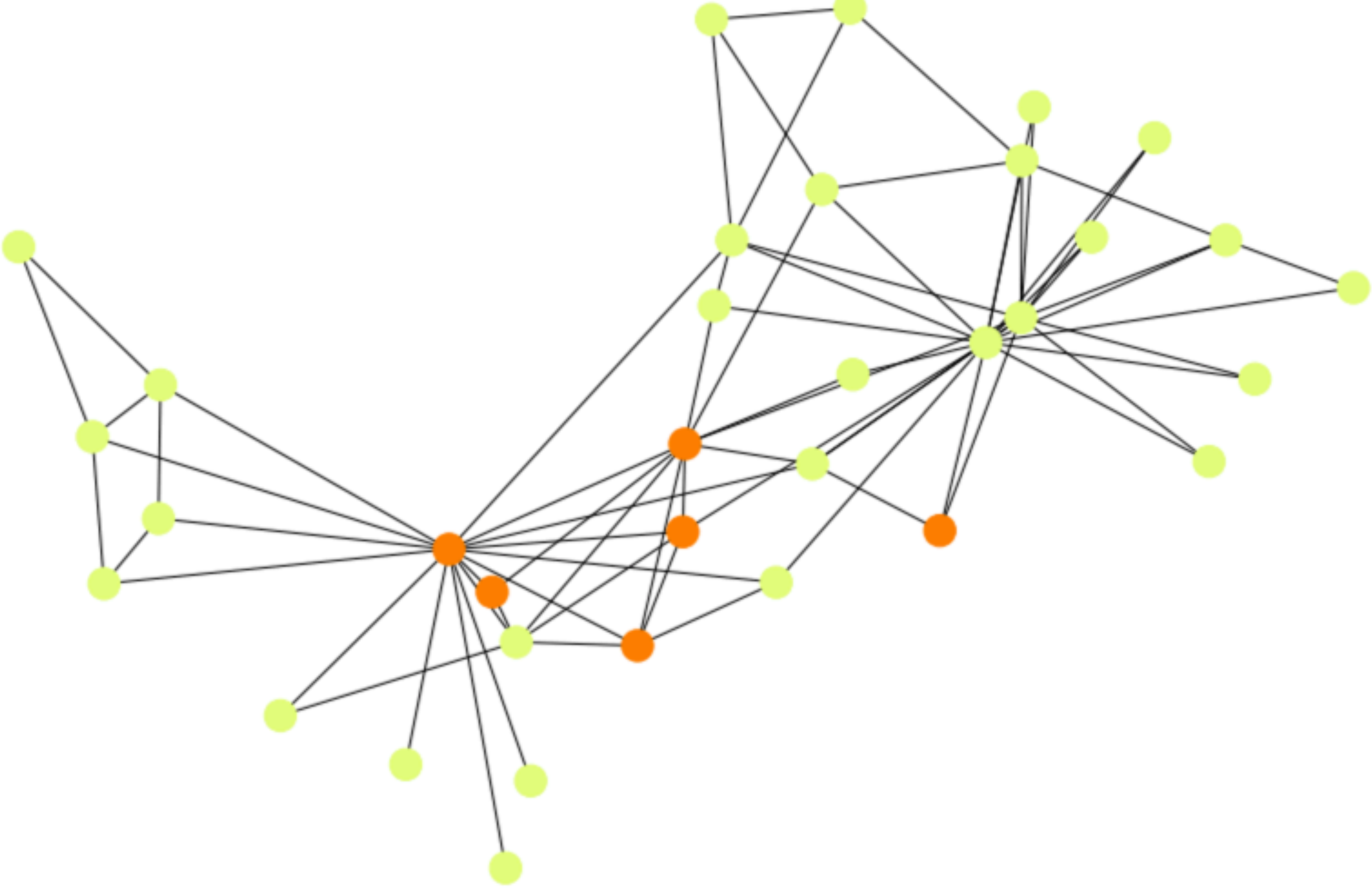
Global Methods

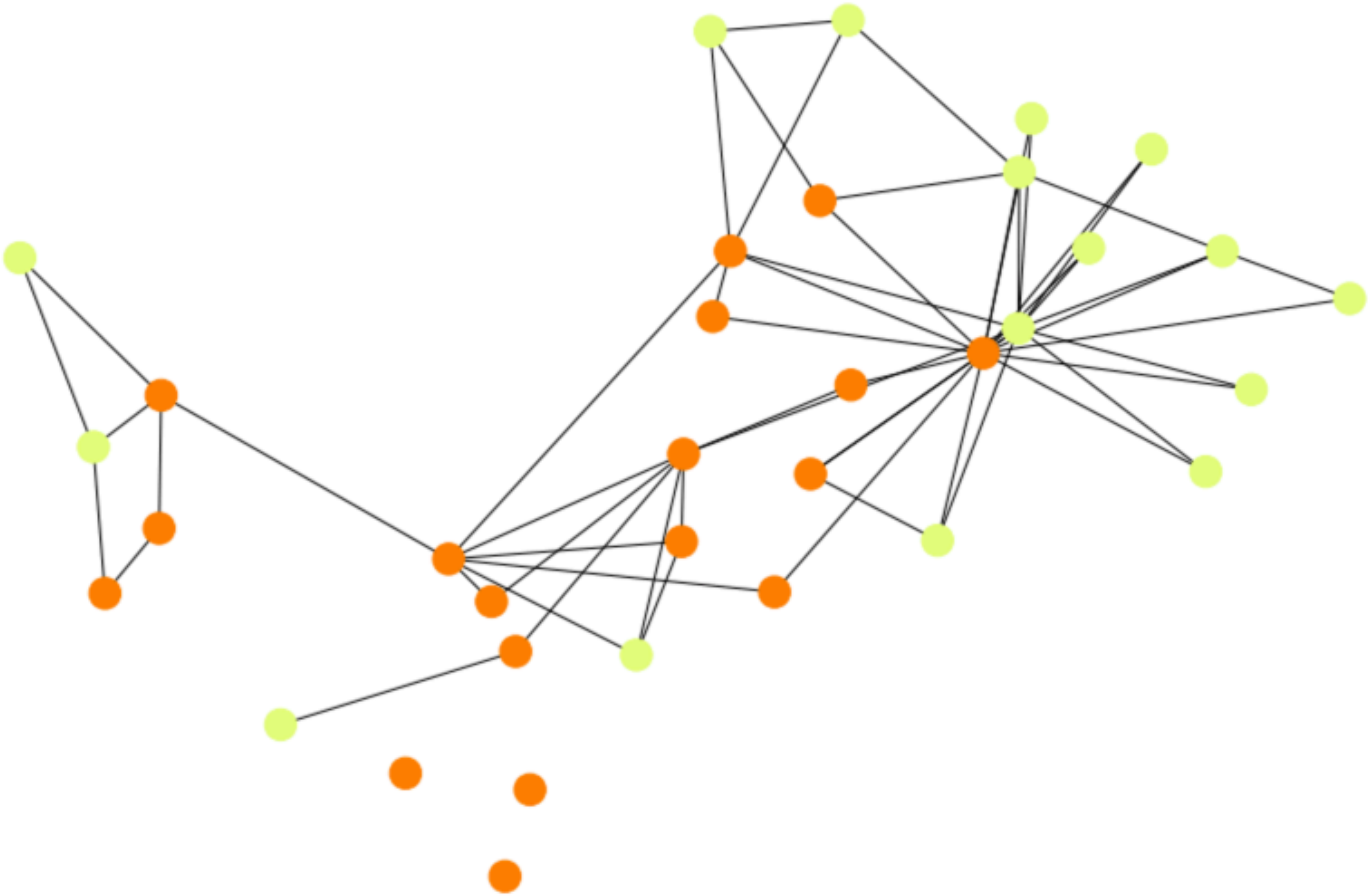
- Sampling-based methods
  - Visual approaches! (sample and look at scatterplots, etc.)
  - Regression-based methods
  - Variance-based methods

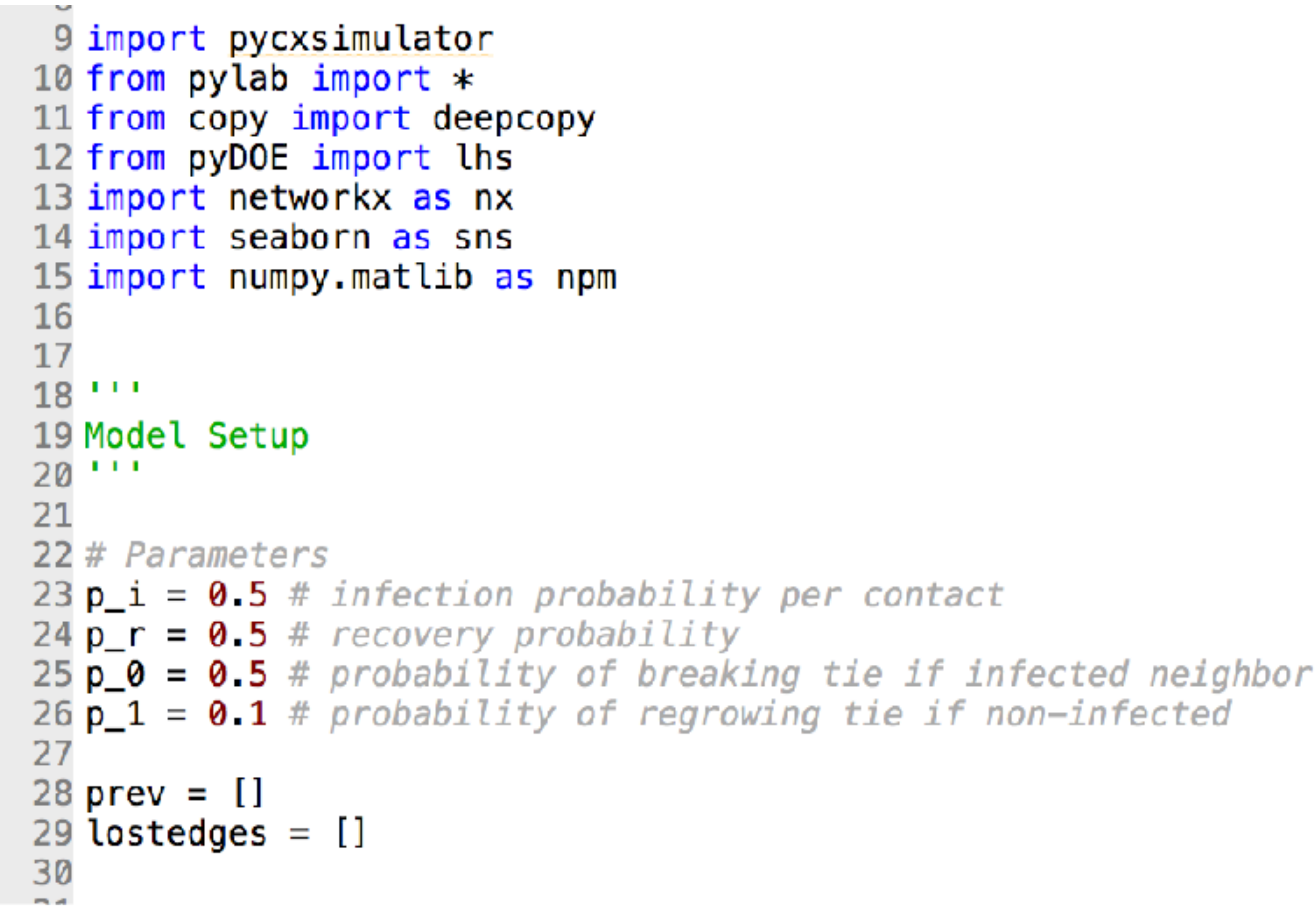
Sampling & visualization

- Sample parameter space, then plot relationships between parameters and model output(s)
- Example: SIR model on the karate club network with tie breaking and regrowth



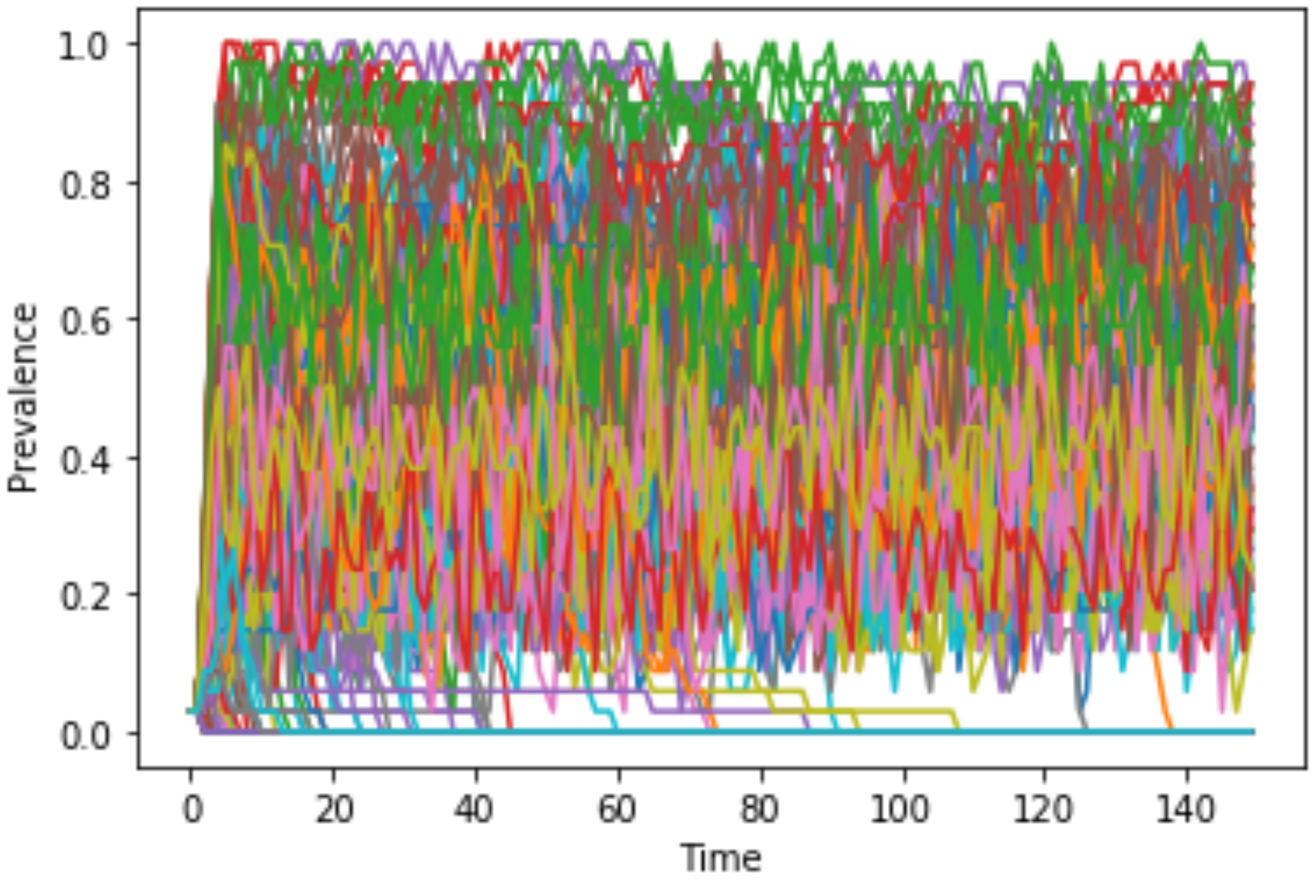


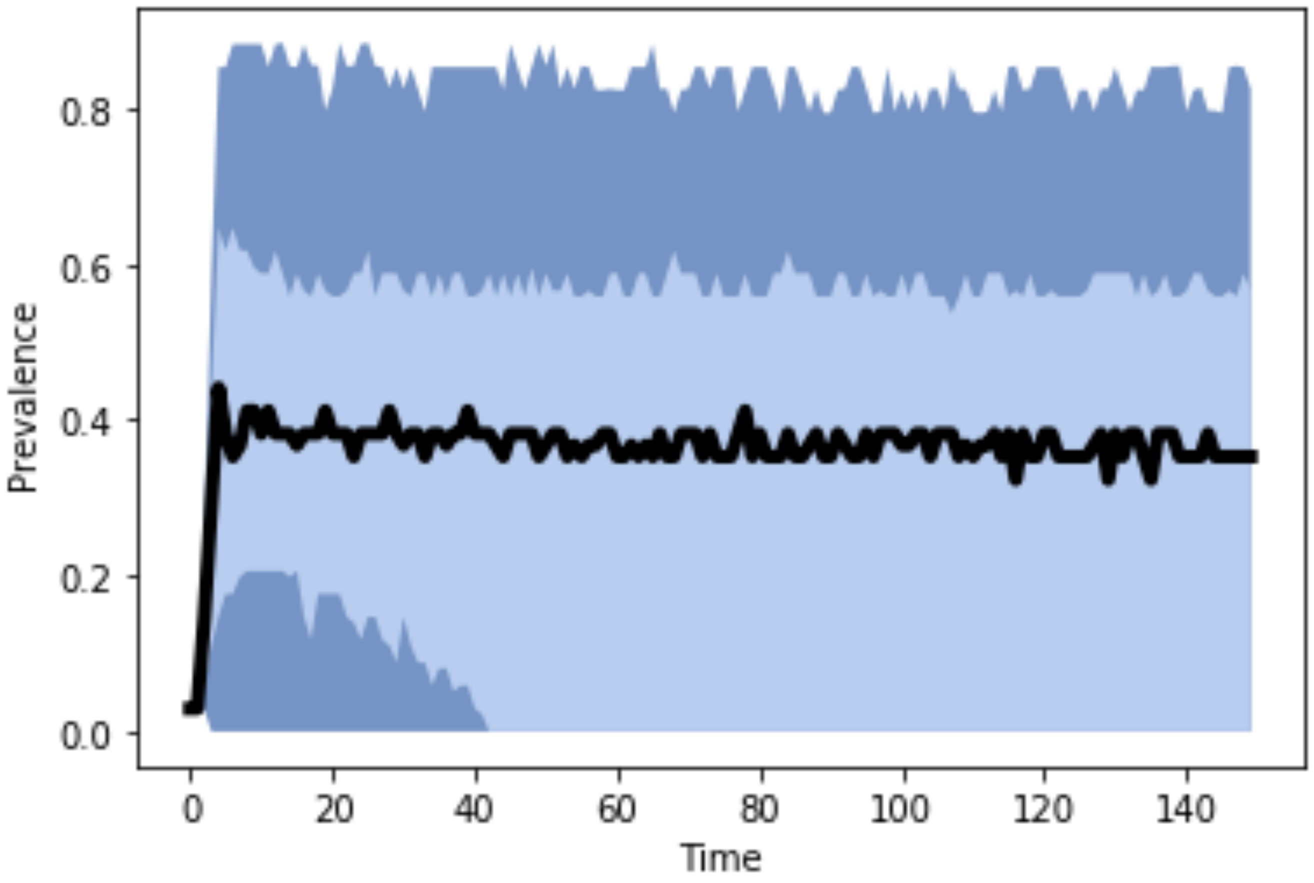


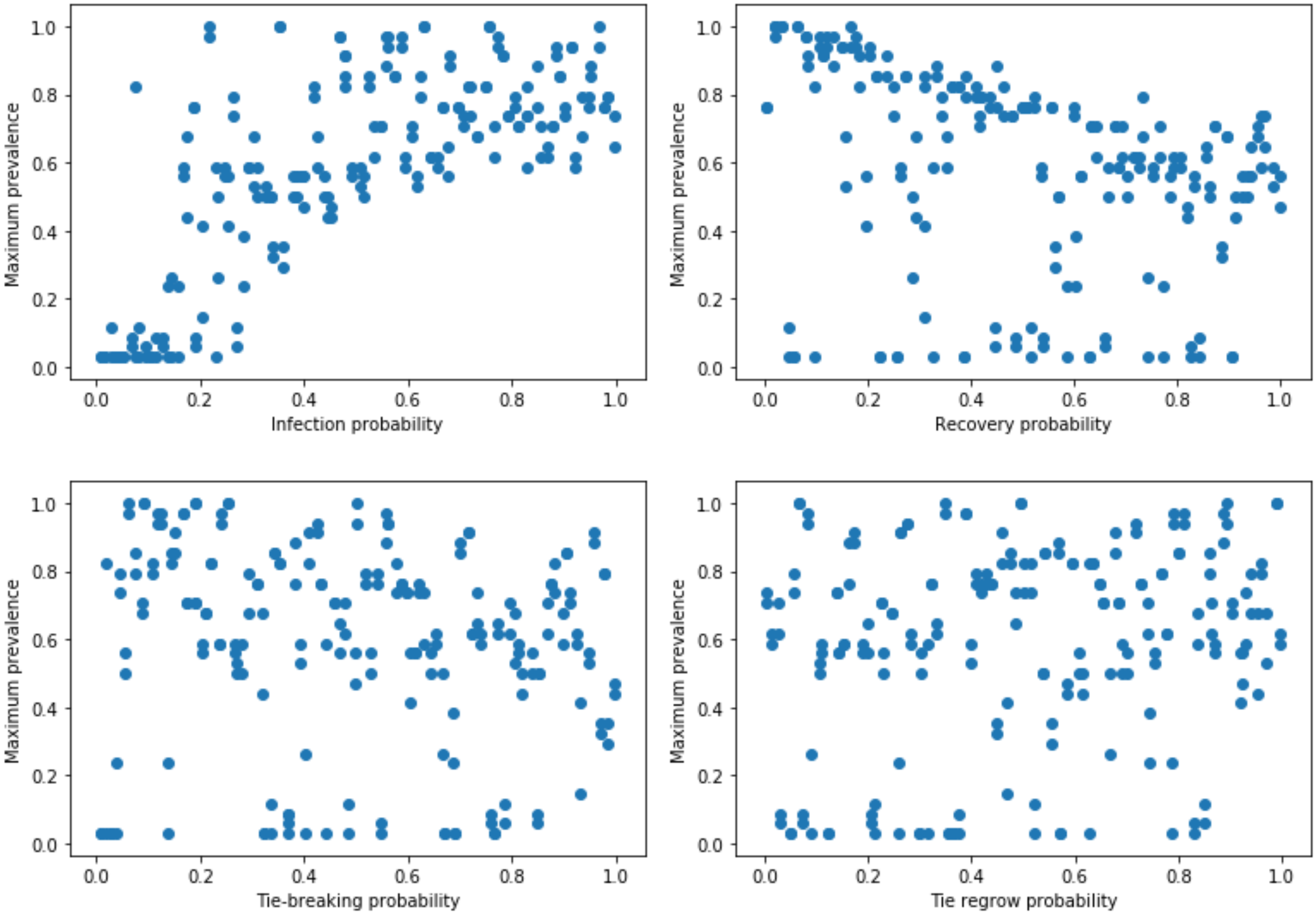


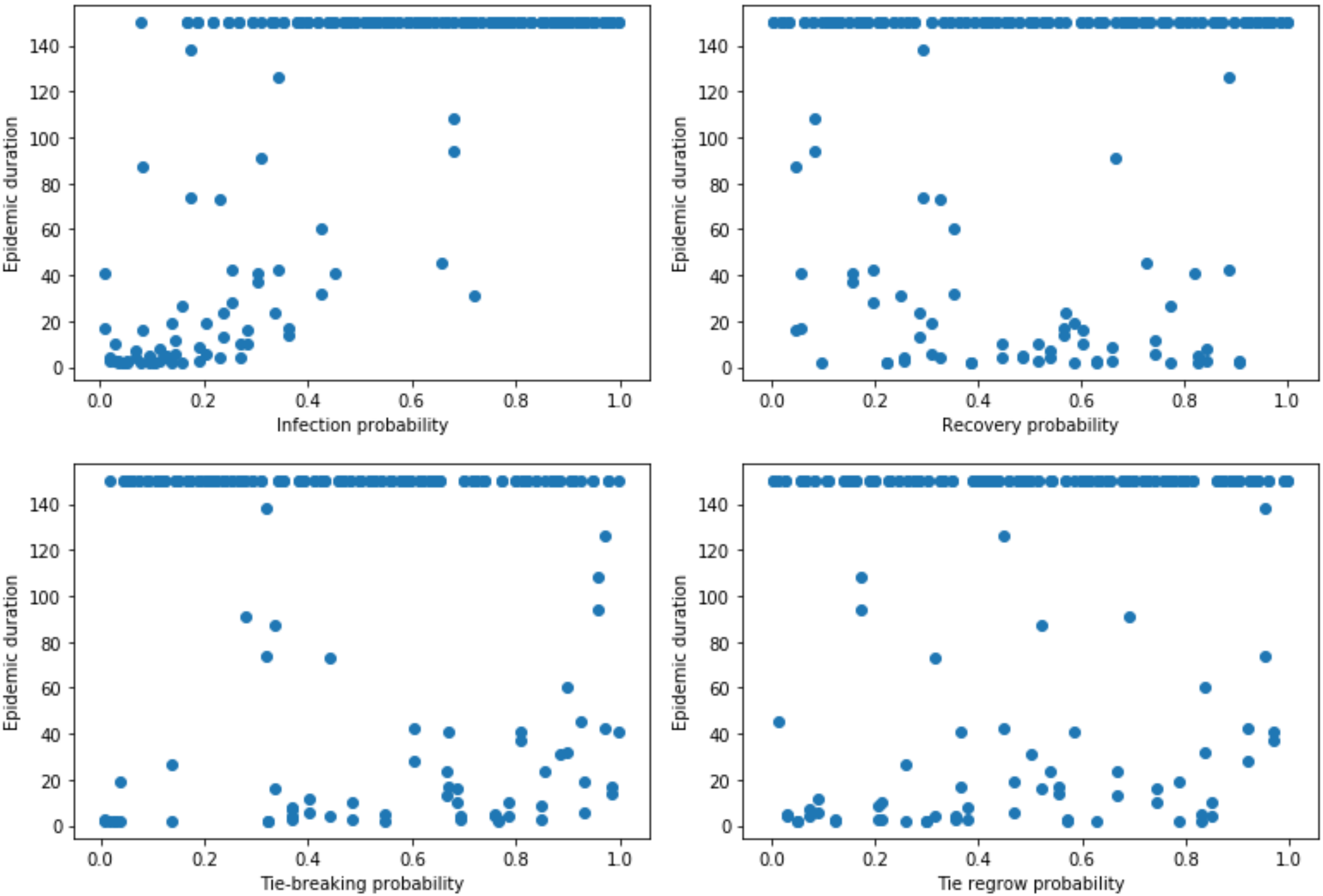
```python
import pycxsimulator
from pylab import *
from copy import deepcopy
from pyDOE import lhs
import networkx as nx
import seaborn as sns
import numpy.matlib as npm


'''
Model Setup
'''

# Parameters
p_i = 0.5 # infection probability per contact
p_r = 0.5 # recovery probability
p_0 = 0.5 # probability of breaking tie if infected neighbor
p_1 = 0.1 # probability of regrowing tie if non-infected

prev = []
lostedges = []

```

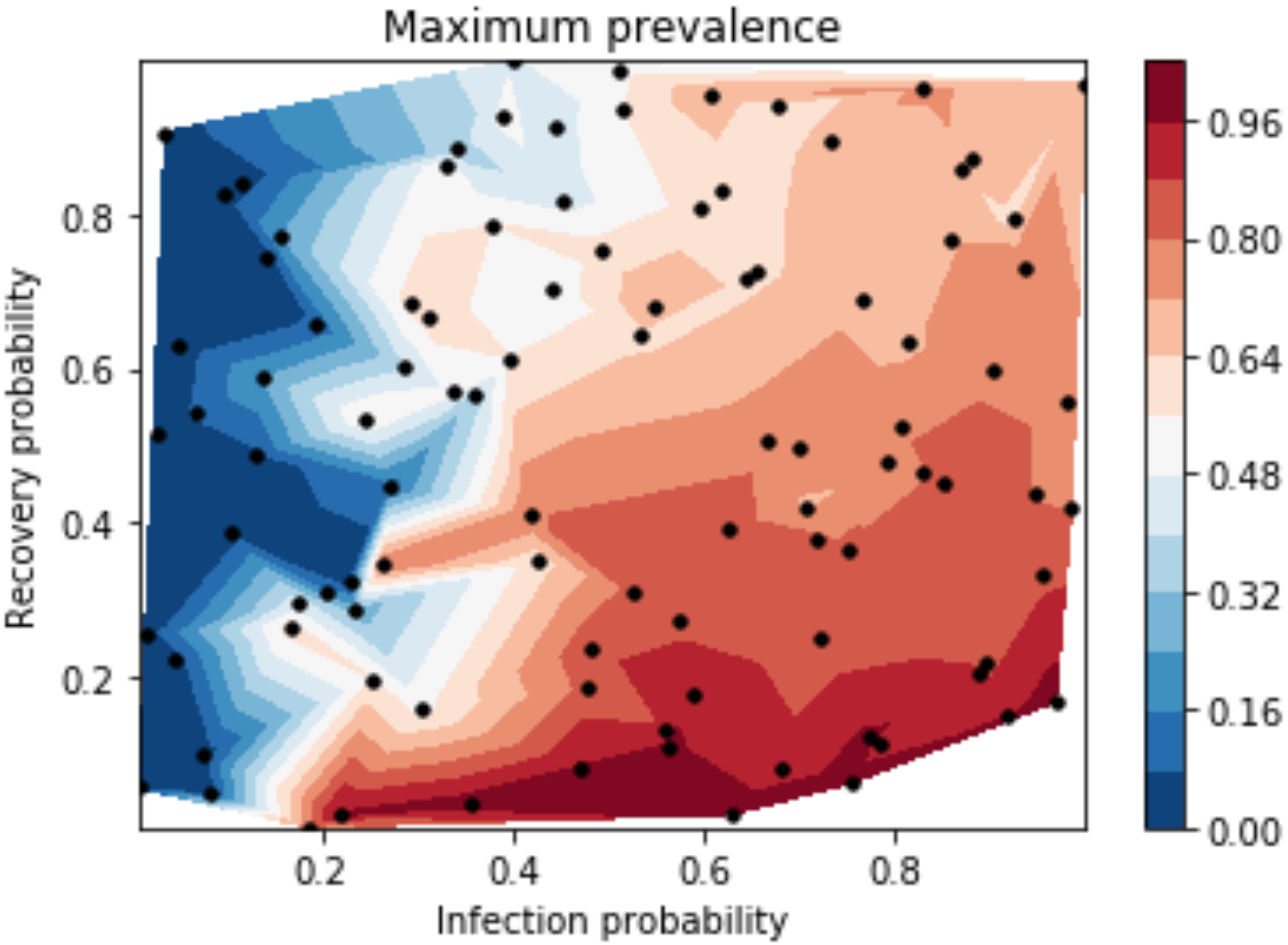
Model dynamics






Maximum prevalence
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Infection probability
Maximum prevalence
Recovery probability
Maximum prevalence
Tie-breaking probability
Maximum prevalence
Tie regrow probability


Epidemic duration
0
20
40
60
80
100
120
140
0.0
0.2
0.4
0.6
0.8
1.0
Infection probability
Recovery probability
Tie-breaking probability
Tie regrow probability


Maximum prevalence
Recovery probability
0.2
0.4
0.6
0.8
Infection probability
0.2
0.4
0.6
0.8
0.00
0.16
0.32
0.48
0.64
0.80
0.96

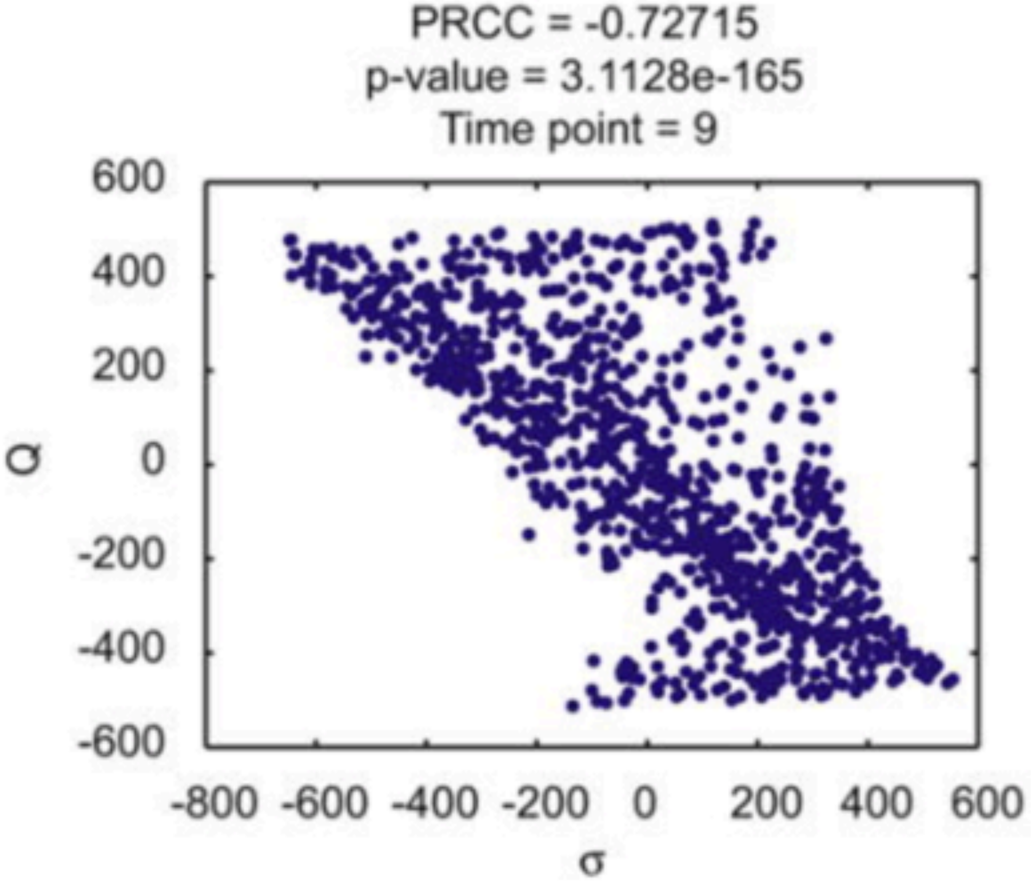
Regression based methods

- Fit linear trend to the data
- Pearson correlation coefficient - correlate parameter and output
  - However, only works for linear relationships
- For nonlinear but monotonic relationships, rank-based correlation coefficients are often useful (draw example)

Partial rank correlation coefficient

- A rank correlation coefficient that accounts for the effects of the other parameters
- Requires monotonic relationship with the output

Partial Rank Correlation Coefficient (PRCC)
PRCC(X_R, Y_R)



https://www.ncbi.nlm.nih.gov/pmc/articles/PMC2570191/

Variance-based methods

- Decomposition of variance (also called the Sobol method)

- Determines how much fo the variance in output is due to each parameter

- Analogous to an ANOVA

- Direct calculation

- Faster options, e.g. eFAST

Word of caution: interpreting statistical results on model outputs

- What does the p-value on one of these regression or variance-based statistics mean?
- What is the source of the uncertainty in these estimates?
- We will discuss this more later

Other Global Methods

- Many other approaches—using PCA, clustering methods, etc.

- Active nonlinear testing—different goal, instead of trying to capture variation, search for extremes, ‘break points’

Active nonlinear tests (ANTs)

- Different goal—instead of trying to capture variation in output/behavior, ANTs try to find extremes or ‘break points’ where behavior change
    - Search for ‘break points’ where behavior changes
- Similar in some ways to a bifurcation analysis
- Doesn't give a sense of what things are more likely, but more a sense of what things are possible

Miller 1998 - https://sfi-edu.s3.amazonaws.com/sfi-edu/production/uploads/sfi-com/dev/uploads/filer/08/15/0815718f-06ca-47d7-bbc6-61ed00511c91/96-03-011.pdf

Active nonlinear tests (ANTs)

- Main idea: use an optimization method to maximize the deviation from the original output
- Find maximally ‘different’ behavior from the original
- Depending on your definition of ‘deviation’ and output, can search for alternate versions of ‘different’
- Wide range of different optimization approaches

Miller 1998 - https://sfi-edu.s3.amazonaws.com/sfi-edu/production/uploads/sfi-com/dev/uploads/filer/08/15/0815718f-06ca-47d7-bbc6-61ed00511c91/96-03-011.pdf

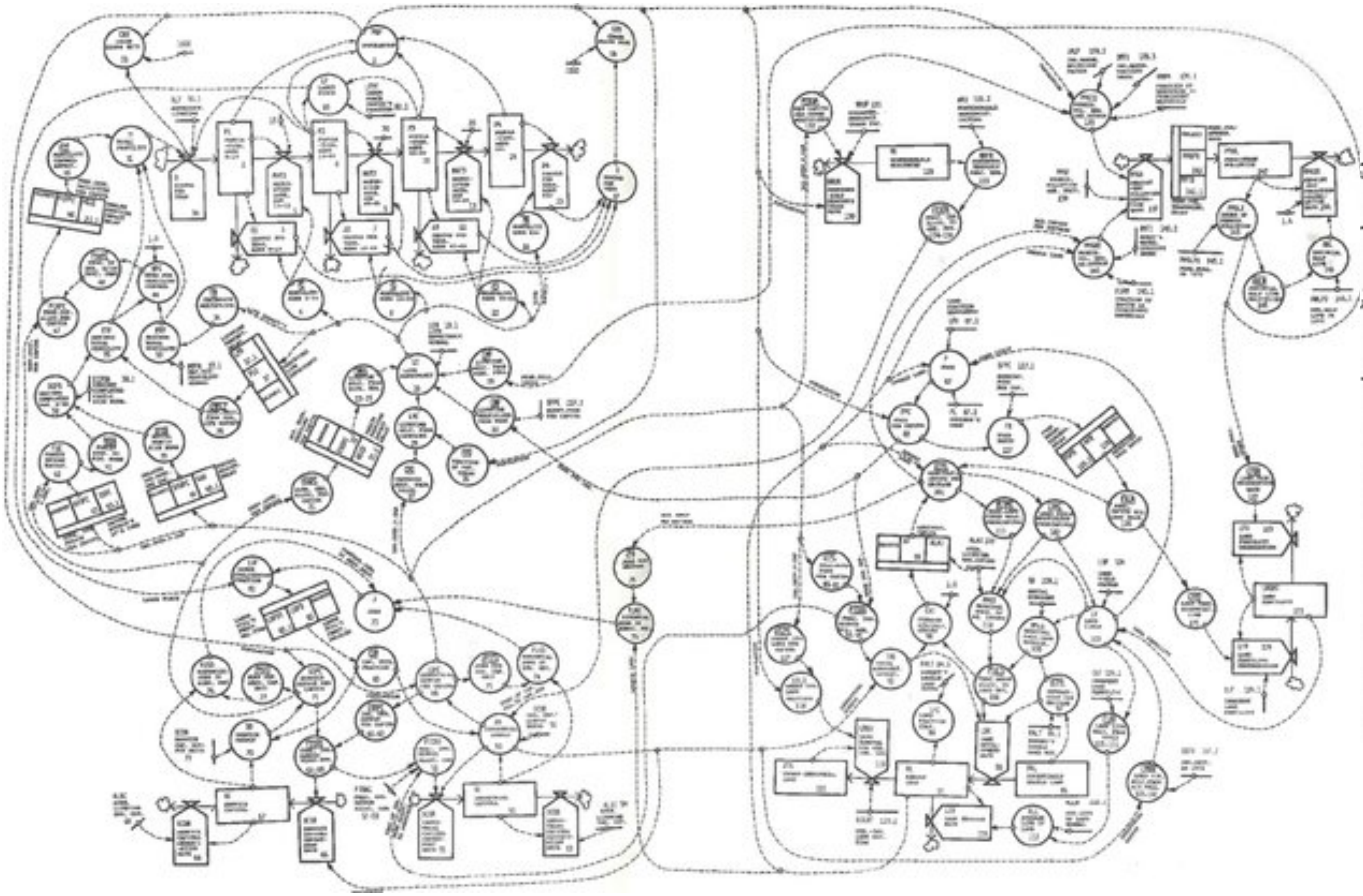
Example: World3

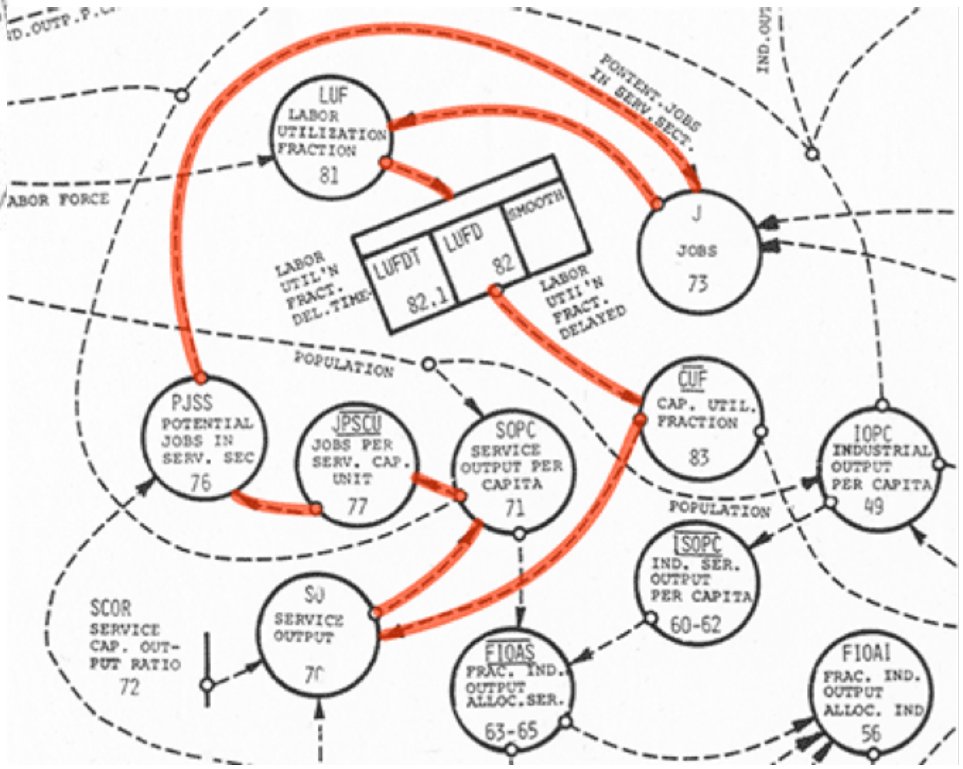
- Ambitious, complex global system model developed in 1974 (Note - NOT an ABM)

- Designed to simulate system dynamics relevant to human sustainability

- Model schematics:
  - 272 model variables with 96 that need to be initialized
  - 150 equations
  - 508 parameters needed to specify functional forms

Example: World3





Example: World3

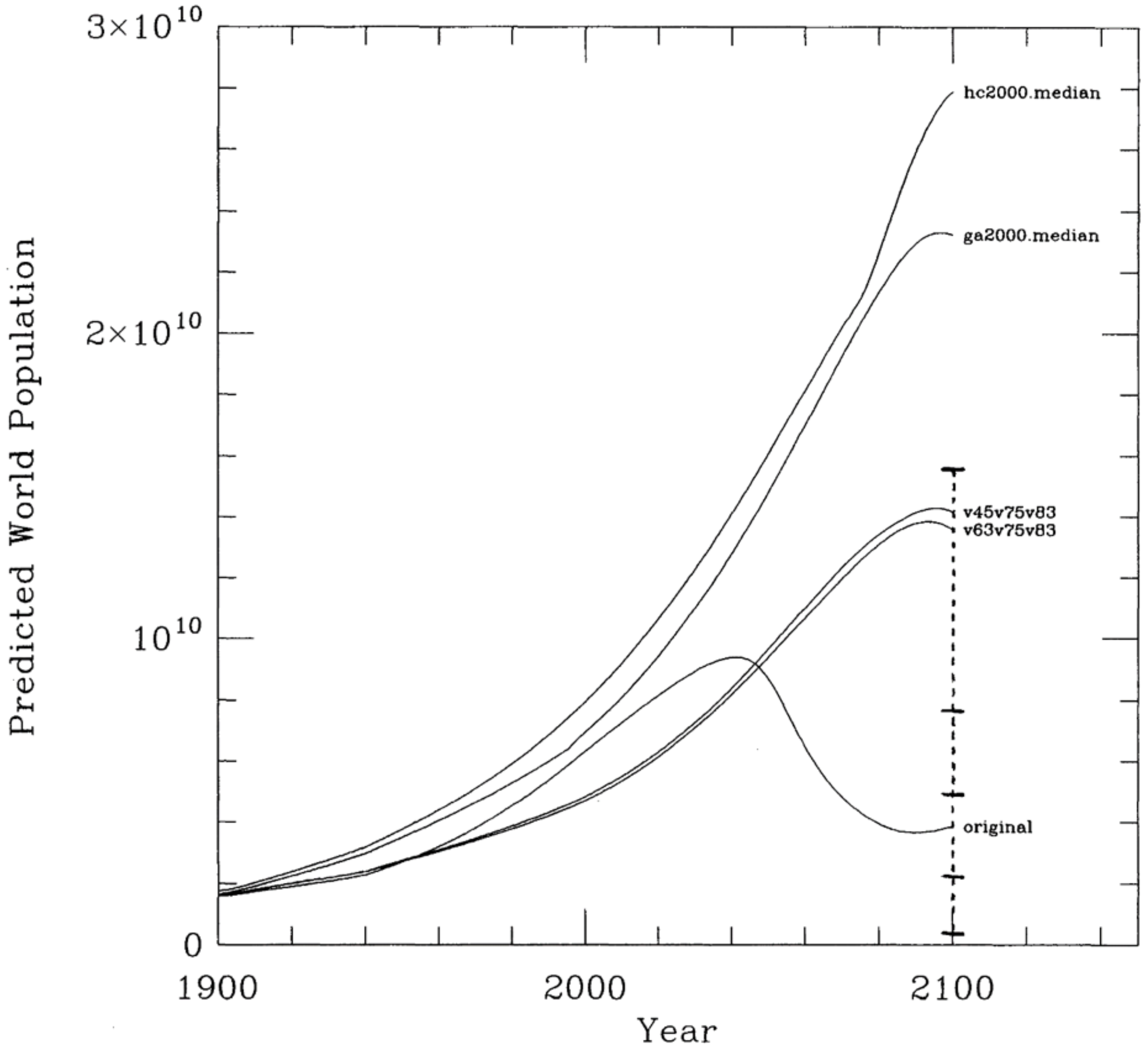
- ANT 1: Maximize estimated pop. in 2100
- ANT 2: minimize pop. peak while conforming to original pre-2000 estimates (i.e. cost function includes a penalty for deviating from pre-2000 estimates)

Example: World3

- Optimization algorithms:
- For each ANT, 2 different optimization algorithms were independently applied in 30 trials for the sake of comparison
    - Hill-climbing algorithm
    - Genetic algorithm
- A random-search of the parameter space was also done to provide a baseline of comparison

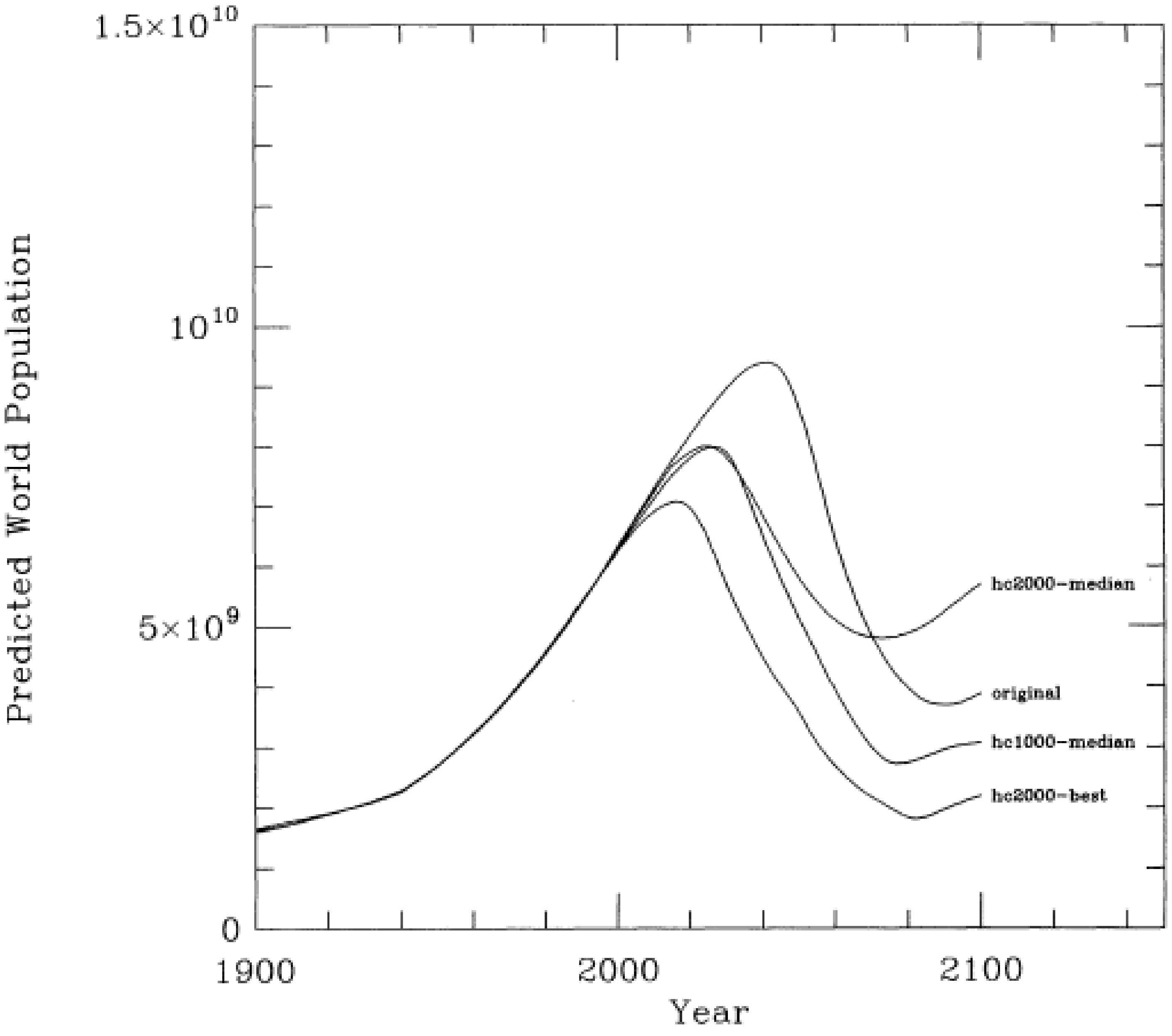
Example: World3

- Results: ANT 1
- Can substantially increase population estimate in 2100 (can “break” the model)



Example: World3

- Results: ANT 3
- Possible to notably reduce population prediction, even with pre-2000 constraint



Active nonlinear tests (ANTs)

- Note that most ANTs do not mean the resulting parameter values are the only ones that will generate a similar deviation from the original

- Nor that other parameter sets with similar deviation values will look anything like the one or ones the optimization algorithm finds!

- In most cases, consider these examples showing that the model can do the behaviors found in the ANT

Dimension reduction & parameter selection

- Fixing insensitive parameters
- Parameter subset selection methods
    - Find subsets of parameters or potentially new parameter combinations that explain most of the behavior (by fixing parameters or parameter combinations that are insensitive)

Surrogate models

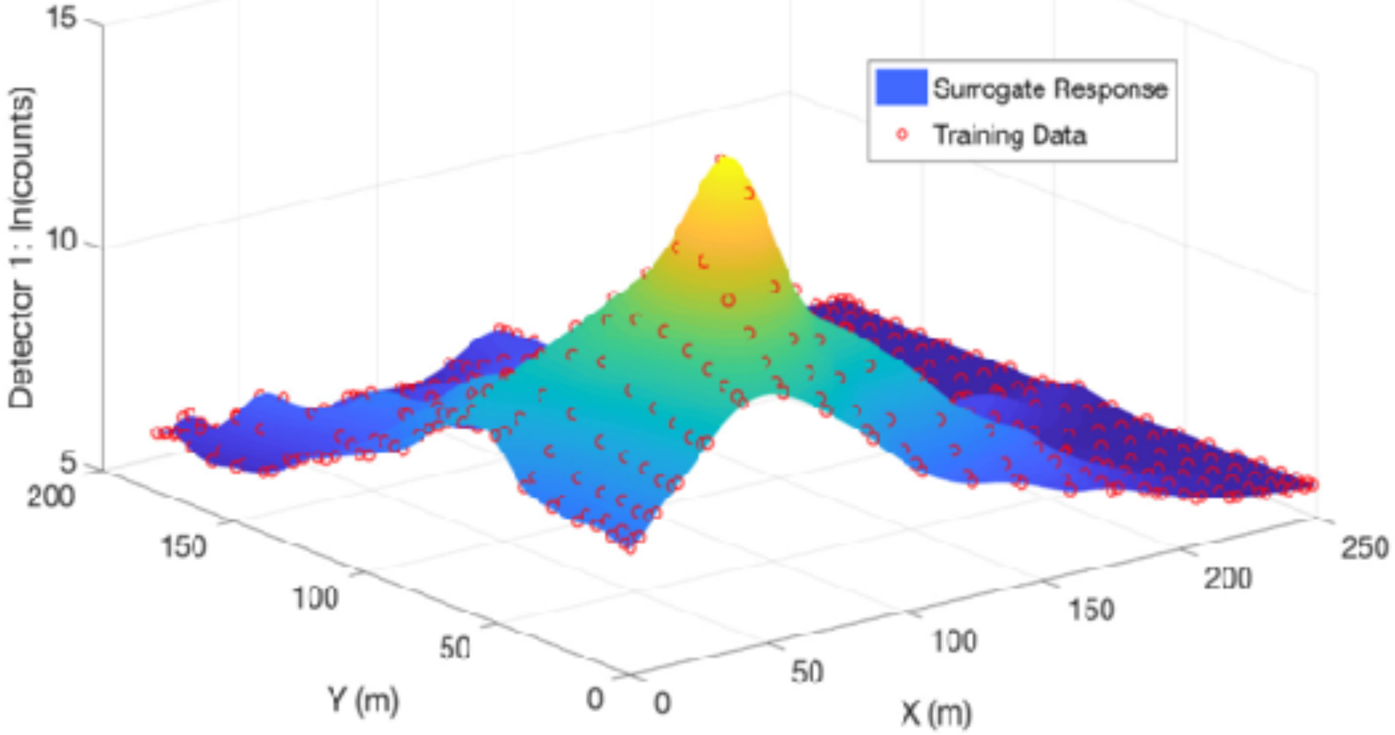
- Global sensitivity methods can be highly computationally expensive—many ABMs take too long to run to be feasible with the number of samples needed to explore space

- ANTs can help, but don’t give a sense of frequency/distribution of outputs/behavior

- Surrogate models (also called emulators, response surfaces) provide another option

Surrogate models

- Idea is to fit a surface or function to the model output(s) as a function of the parameters
    - Choose a functional form that is cheap to evaluate many times (e.g. polynomial, linear)



Cook JA, Smith RC, Hite JM, Stefanescu R, Mattingly J. Application and Evaluation of Surrogate Models for Radiation Source Search. Algorithms. 2019 Dec;12(12):269.

Surrogate models

- Use a smaller number of points to fit the surface, then sample a large number of points to run sensitivity analysis

- Re-run using the true model on regions of interest

For next time…

- Readings
  - Marino, Simeone, et al. "A methodology for performing global uncertainty and sensitivity analysis in systems biology." Journal of theoretical biology 254.1 (2008): 178-196.
  - Additional papers on the website